Development of Particle Scattering Models for Supporting Radiative Transfer Simulations

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Acknowledgement: Ping Yang, George Kattawar, Michael Mishchenko, Fuzhong Weng

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Critical Issues

- **Tools**: Accurate and efficient computational methods to compute the optical properties of arbitrary shaped and inhomogeneous particles
- **Physics**: Physical mechanism on how particle microphysics affects electromagnetic wave scattering and possible influence on radiative transfer
- **Models**: Develop “suitable” models for ice crystals and aerosols for atmospheric radiative transfer simulations
Maxwell's Equations

\[ \nabla \times \vec{H}(\vec{r}, t) = \frac{\varepsilon}{c} \frac{\partial \vec{E}}{\partial t}, \]

\[ \nabla \times \vec{E}(\vec{r}, t) = -\frac{\mu}{c} \frac{\partial \vec{H}}{\partial t} \]

Classical theory of light.

James Clerk Maxwell
Far-field Scattering

\[ k(r - a) >> 1; \quad r >> a; \quad r >> ka^2 / 2 \]

- Amplitude Scattering Matrix

\[
\begin{pmatrix}
E_{||}^S \\
E_{\perp}^S
\end{pmatrix} = \frac{e^{ikr}}{-ikr} \begin{pmatrix}
S_2 & S_3 \\
S_4 & S_1
\end{pmatrix}
\begin{pmatrix}
E_{||}^i \\
E_{\perp}^i
\end{pmatrix}
\]

scattered spherical wave
## Computational Progress

<table>
<thead>
<tr>
<th></th>
<th>Invariant Imbedding T-matrix (Bi et al., 2013; Bi and Yang, 2014)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Physical-geometric optics hybrid (Bi et al., 2011, Bi et al., 2012)</td>
</tr>
<tr>
<td>3</td>
<td>Debye’s series, complex angular momentum (Bi et al., 2013; Bi et al., 2017; Bi et al., 2019)</td>
</tr>
</tbody>
</table>
Invariant Imbedding T-matrix

Distribution of refractive indices

Nonspherical particle = inhomogeneous sphere

\[ T_{mnmn}(r + dr) = Q_{11}^m(r + dr) + \left[ I + Q_{12}^m(r + dr) \right] \left[ I - T_{mnmn}(r)Q_{22}^m(r + dr) \right]^{-1} T_{mnmn}(r) \left[ I + \tilde{Q}_{12}^m(r + dr) \right] \]

Johnson (1988), Bi et al. (2013, 2014)
Arbitrarily Shaped Particles

An inhomogeneous sphere Discretize to multi-layered sphere Non-unity refractive index of each spherical surface

Features

• Arbitrary shaped and inhomogeneous particles
• Analytical random orientation average
• Particle size parameter upto geometric-optics domains
Physical geometric-optics Hybrid

\[ E_{\text{sc}}(\vec{r}) = \frac{\exp(ikr) - ikr^3}{-ikr^4} \int \int \int_v \left[ \hat{m}^2(\vec{r}') - 1 \right] \left[ \vec{E}(\vec{r}') - \hat{r} \left( \vec{r} \cdot \vec{E}(\vec{r}') \right) \right] \exp(-ik\vec{r} \cdot \vec{r}') d^3\vec{r}' \]

Debye's Series

http://www.philiplaven.com/p2-1.html
T-matrix Expansion: Debye’s Series

\[ T = -\frac{1}{2} \left[ 1 - R_{11} - T_{12} \left( \sum_{n=0}^{\infty} (R_{22})^n \right) T_{21} \right] = -\frac{1}{2} \left[ 1 - R_{11} - T_{12} \frac{1}{1 - R_{22}} T_{21} \right] \]

Unlike ray-tracing, exact decomposition; Understanding physical mechanism

Validated for Convex Nonspherical Particles

(Debye, 1908; Van der Pol and H. Bremmer, 1937; Feng et al, 2010)

Lei Bi, Feng Xu, Gérard Gouesbet, 2018; Phys. Rev. A 98,053809; to be submitted, 2019
Ice Crystal Optical Property Database

Yang, P., L. Bi, B. A. Baum, K. N. Liou, G. W. Kattawar, M. I. Mishchenko, and B. Cole, 2013: Spectrally consistent scattering, absorption, and polarization properties of atmospheric ice crystals at wavelengths from 0.2 to 100 μm. Journal of the Atmospheric Sciences, 70, 330-347.


Major Aerosol Types

Sulfate
Carbonaceous
Dust
Sea salt

NASA | GEOS-5 Aerosols
LiDAR Observations

- LiDAR observations are sensitive to the back-scattering optical properties of a volume element.
- Interpretation: understanding LiDAR signals
- Model constraints: “suitable” models for general radiative transfer simulations
Phase Matrix and Depolarization Ratio

\[
\begin{pmatrix}
I_s \\
Q_s \\
U_s \\
V_s
\end{pmatrix} = \frac{1}{k^2 r^2} \begin{pmatrix}
P_{11} & P_{12} & 0 & 0 \\
P_{12} & P_{22} & 0 & 0 \\
0 & 0 & P_{33} & P_{34} \\
0 & 0 & -P_{43} & P_{44}
\end{pmatrix} \begin{pmatrix}
I_i \\
Q_i \\
U_i \\
V_i
\end{pmatrix}
\]

**Assumption:** incident light is 100% polarized parallel to the scattering plane

\[
\begin{pmatrix}
I_i \\
Q_s = (P_{12} + P_{22}) I_i \\
0 \\
0
\end{pmatrix} \rightarrow \begin{pmatrix}
I_i \\
Q_s \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
(P_{11} + P_{12}) I_i \\
(P_{12} + P_{22}) (I_i - I_{s\perp}) \\
0 \\
P_{11} + 2P_{12} + P_{22}
\end{pmatrix}
\]

**Result:** scattered light is in general partially polarized, i.e. the incident light is depolarized
Linear Depolarization Ratio (Random Orientation)

\[
\delta(180^\circ) = \frac{1 - P_{22}/P_{11}}{1 + P_{22}/P_{11}}
\]

\[0 \leq \delta \leq 1\]

Zero depolarization ratio cases:

Single scattering by water droplets in water clouds

Nonspherical water droplets in rain with axial incidence
Incident Beam

Backscattered Beam

$\delta(\theta = 180^\circ) = 100\%$
Size Distribution

\[
\delta = \frac{1 - \int_{r_{\text{min}}}^{r_{\text{max}}} P_{22,180^\circ} (r) C_{\text{sca}} (r) \frac{dN}{dr} \, dr}{\int_{r_{\text{min}}}^{r_{\text{max}}} P_{11,180^\circ} (r) C_{\text{sca}} (r) \frac{dN}{dr} \, dr} \frac{1 + \int_{r_{\text{min}}}^{r_{\text{max}}} P_{22,180^\circ} (r) C_{\text{sca}} (r) \frac{dN}{dr} \, dr}{\int_{r_{\text{min}}}^{r_{\text{max}}} P_{11,180^\circ} C_{\text{sca}} (r) \frac{dN}{dr} \, dr}
\]

\[
\frac{dN}{dr} = \frac{N_0}{\sqrt{2\pi r \log(\sigma) \ln(10)}} \exp \left\{ -\frac{1}{2} \left[ \frac{\log(r) - \log(r_m)}{\log(\sigma)} \right]^2 \right\}
\]
Backscatter Lidar Ratio

\[
S = \frac{\int_{r_{\text{min}}}^{r_{\text{max}}} C_{\text{ext}}(r) \frac{dN}{dr} \, dr}{P_{11,180^\circ} \int_{r_{\text{min}}}^{r_{\text{max}}} C_{\text{sca}}(r) \frac{dN}{dr} \, dr}
\]
How to model particle shapes?
Super-spheroidal Space

\[
\left(\frac{x}{a}\right)^\frac{2}{n} + \left(\frac{y}{a}\right)^\frac{2}{n} + \left(\frac{z}{c}\right)^\frac{2}{n} = 1
\]

Feldspar

Volcanic ash

Sea salt

Sodium chloride


Nearly spherical Particles show quite different depolarization features in aerosol refractive index regime
Spheroids, Refractive Index: $1.55 + i0.003$

Depolarization Ratio ( % )

First reported by Mishchenko, M. I., and J. W. Hovenier (1995) at this refractive index

>60%
Bi et al., Optics Express, 2018
Different Physical Origins obtained from Debye’s series

Waves after 1 internal reflection

Interference

Lei Bi, Feng Xu, Gérard Gouesbet, 2018; Phys. Rev. A 98,053809;
CALIPSO DATA (2012)

Mix2-enhanced: STS + high number densities/volumes of Nitric Acid Trihydrate (NAT) particles

Polar Stratospheric Cloud (PSC) detection described in (Pitts et al, 2009)
Dust optical modeling
Why more freedom is useful?

S = average projected area
V = volume of a particle

\[ SI = \frac{3V}{4\pi \left( \frac{S}{\pi} \right)^{3/2}} \]

SI = 1 for a sphere
Advantage of SI

• If two nonspherical particles have the same shape index (SI), they have identical volume and projected area.
Stereogrammetry

SI = 0.67

SI = 0.82

volume, projected area, aspect ratio

(Image from Lindqvist, et al., 2014)
Comparison with Laboratory Measurements
Table 1

<table>
<thead>
<tr>
<th>Sample</th>
<th>$r_{\text{eff}}/\mu\text{m}$</th>
<th>$v_{\text{eff}}$</th>
<th>Re(m)</th>
<th>Im(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allende</td>
<td>0.8</td>
<td>3.3</td>
<td>1.65</td>
<td>0.001</td>
</tr>
<tr>
<td>Feldspar*</td>
<td>1.0</td>
<td>1.0</td>
<td>1.50–1.60</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Red clay*</td>
<td>1.5</td>
<td>1.6</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Green clay*</td>
<td>1.55</td>
<td>1.4</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Quartz</td>
<td>2.3</td>
<td>2.3</td>
<td>1.54</td>
<td>0</td>
</tr>
<tr>
<td>Martian analog palagonite</td>
<td>4.5</td>
<td>7.3</td>
<td>1.50</td>
<td>0.001–0.0001</td>
</tr>
<tr>
<td>Loess*</td>
<td>3.9</td>
<td>2.6</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Sahara*</td>
<td>8.2</td>
<td>4.0</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Forsterite (initial)</td>
<td>1.3</td>
<td>5.4</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Forsterite (small)</td>
<td>1.3</td>
<td>3.1</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Forsterite (washed)</td>
<td>4.7</td>
<td>1.8</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Olivine (S)</td>
<td>1.3</td>
<td>5.4</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Olivine (M)</td>
<td>2.6</td>
<td>5.0</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Olivine (L)</td>
<td>6.3</td>
<td>6.8</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Olivine (XL)</td>
<td>6.3</td>
<td>6.8</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Volcanic ash (E. Chichon)</td>
<td>3.2</td>
<td>1.4</td>
<td>1.50–1.70</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Volcanic ash (Ph. atubo)</td>
<td>3.6</td>
<td>12.3</td>
<td>1.50–1.70</td>
<td>0.001–0.0001</td>
</tr>
<tr>
<td>Volcanic ash (Lokon)</td>
<td>7.1</td>
<td>2.6</td>
<td>1.50–1.60</td>
<td>0.001–0.00001</td>
</tr>
<tr>
<td>Volcanic ash (Mnt. St. Helens)</td>
<td>4.1</td>
<td>9.5</td>
<td>1.48–1.56</td>
<td>0.0018</td>
</tr>
<tr>
<td>Volcanic ash (Redoubt A)</td>
<td>4.1</td>
<td>9.7</td>
<td>1.48–1.56</td>
<td>0.0018</td>
</tr>
<tr>
<td>Volcanic ash (Redoubt B)</td>
<td>6.4</td>
<td>7.6</td>
<td>1.48–1.56</td>
<td>0.0018</td>
</tr>
<tr>
<td>Volcanic ash (Spurr Ashton)</td>
<td>2.7</td>
<td>4.9</td>
<td>1.48–1.56</td>
<td>0.0018–0.02</td>
</tr>
<tr>
<td>Volcanic ash (Spurr Gunsight)</td>
<td>3.5</td>
<td>8.2</td>
<td>1.48–1.56</td>
<td>0.0018–0.02</td>
</tr>
<tr>
<td>Volcanic ash (Spurr Anchorage)</td>
<td>4.8</td>
<td>8.8</td>
<td>1.48–1.56</td>
<td>0.0018–0.02</td>
</tr>
<tr>
<td>Volcanic ash (Spurr Stop 33)</td>
<td>14.4</td>
<td>6.6</td>
<td>1.48–1.56</td>
<td>0.0018–0.02</td>
</tr>
</tbody>
</table>

Note. The $r_{\text{eff}}$ is the effective radius; $v_{\text{eff}}$ is the variance; Re(m) and Im(m) are the real part and the imaginary part of the estimated refractive index, respectively.

*These samples have been investigated in Merikallio et al., (2011).
Volcanic ash (Pinatubo)  
m=1.60+i0.001
Three key scattering matrix elements ($P_{11}$, $-P_{12}/P_{11}$, and $P_{22}/P_{11}$) of simulations and measurements at 632.8nm wavelength for five volcanic ash samples. The refractive index was selected to be 1.60+i0.001 for all the samples.
# LiDAR DUST Observations

<table>
<thead>
<tr>
<th>Filed campaign</th>
<th>time</th>
<th>location</th>
</tr>
</thead>
<tbody>
<tr>
<td>SAMUM (Saharan Mineral Dust Experiment)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SAMUM I</td>
<td>May–June 2006</td>
<td>southern Morocco</td>
</tr>
<tr>
<td>SAMUM-2a</td>
<td>January–February 2008</td>
<td>Cape Verde</td>
</tr>
<tr>
<td>SAMUM-2b</td>
<td>May–June 2008</td>
<td></td>
</tr>
<tr>
<td>SALTRACE-3 (Saharan Aerosol Longrange Transport and Aerosol-Cloud-Interaction Experiment)</td>
<td>June–July 2014</td>
<td>Caribbean island of Barbados</td>
</tr>
<tr>
<td>SHADOW campaign (Study of Saharan Dust Over West Africa)</td>
<td>March–April 2015</td>
<td>Senegal</td>
</tr>
<tr>
<td>CADEX (Central Asian Dust EXperiment)</td>
<td>March 2015 to August 2016</td>
<td>Dushanbe, Tajikistan</td>
</tr>
</tbody>
</table>
**Depolarization ratio observations at different locations**

<table>
<thead>
<tr>
<th>Location</th>
<th>0.355 $\mu m$</th>
<th>0.532 $\mu m$</th>
<th>1.064 $\mu m$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Midwest</td>
<td>0.246 ± 0.018</td>
<td>0.304 ± 0.005</td>
<td>0.270 ± 0.005</td>
<td>Burton et al. (2015)</td>
</tr>
<tr>
<td>Chihuahuan</td>
<td>0.243 ± 0.046</td>
<td>0.373 ± 0.014</td>
<td>0.383 ± 0.006</td>
<td>Burton et al. (2015)</td>
</tr>
<tr>
<td>Caribbean</td>
<td>-</td>
<td>0.327 ± 0.018</td>
<td>0.278 ± 0.012</td>
<td>Burton et al. (2015)</td>
</tr>
<tr>
<td>Pico de Orizaba</td>
<td>-</td>
<td>0.334 ± 0.018</td>
<td>0.400 ± 0.009</td>
<td>Burton et al. (2015)</td>
</tr>
<tr>
<td>Barbados</td>
<td>0.252 ± 0.03</td>
<td>0.28 ± 0.02</td>
<td>0.225 ± 0.02</td>
<td>Haarig et al. (2017)</td>
</tr>
<tr>
<td>Ouarzazate</td>
<td>0.28 ± 0.05</td>
<td>0.31 ± 0.03</td>
<td>0.26 ± 0.04</td>
<td>Freudenthaler et al. (2009)</td>
</tr>
<tr>
<td>Ouarzazate</td>
<td>0.24 ± 0.07</td>
<td>0.30 ± 0.03</td>
<td>0.27 ± 0.04</td>
<td>Freudenthaler et al. (2009)</td>
</tr>
<tr>
<td>Tajikistan</td>
<td>0.18-0.29</td>
<td>0.31-0.35</td>
<td>-</td>
<td>Hofer et al. (2017)</td>
</tr>
<tr>
<td>Cape Verde</td>
<td>0.24-0.27</td>
<td>0.29-0.31</td>
<td>-</td>
<td>Groß et al. (2011)</td>
</tr>
<tr>
<td>Munich</td>
<td>0.30 ± 0.07</td>
<td>0.34 ± 0.03</td>
<td>-</td>
<td>Wiegner et al. (2011)</td>
</tr>
</tbody>
</table>
LiDAR ratio measurements at Different Locations

<table>
<thead>
<tr>
<th>Location</th>
<th>0.355 $\mu$m</th>
<th>0.532 $\mu$m</th>
<th>1.064 $\mu$m</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tajikistan</td>
<td>46.9 ± 2.1</td>
<td>42.9 ± 3.2</td>
<td>-</td>
<td>Hofer et al. (2017)</td>
</tr>
<tr>
<td>Cape Verde</td>
<td>58 ± 7</td>
<td>62 ± 5</td>
<td>-</td>
<td>Groß et al. (2011)</td>
</tr>
<tr>
<td>Munich</td>
<td>58 ± 13</td>
<td>56 ± 10</td>
<td>-</td>
<td>Wiegner et al. (2011)</td>
</tr>
<tr>
<td>Ouarzazate</td>
<td>52.9 ± 7</td>
<td>54.8 ± 6.7</td>
<td>54.9 ± 12.7</td>
<td>Tesche et al. (2009)</td>
</tr>
<tr>
<td>Gwangju</td>
<td>56 ± 10</td>
<td>51 ± 6</td>
<td>-</td>
<td>Noh et al. (2008)</td>
</tr>
<tr>
<td>Mbour</td>
<td>54 ± 8</td>
<td>53 ± 8</td>
<td>-</td>
<td>Veselovskii et al. (2016)</td>
</tr>
<tr>
<td>Gwangju</td>
<td>52 ± 7</td>
<td>53 ± 8</td>
<td>-</td>
<td>Shin et al. (2016)</td>
</tr>
<tr>
<td>Portugal</td>
<td>45 ± 8</td>
<td>53 ± 7</td>
<td>-</td>
<td>Preßler et al. (2011)</td>
</tr>
<tr>
<td>Barbados</td>
<td>53 ± 5</td>
<td>53 ± 7</td>
<td>-</td>
<td>Groß et al. (2015)</td>
</tr>
<tr>
<td>Tokyo</td>
<td>48.6 ± 8.5</td>
<td>43.1 ± 7.0</td>
<td>-</td>
<td>Murayama et al. (2004)</td>
</tr>
<tr>
<td>Guangzhou</td>
<td>48.5 ± 7.5</td>
<td>51.7 ± 8.3</td>
<td>-</td>
<td>Heese et al. (2017)</td>
</tr>
</tbody>
</table>
# Laboratory Measurements

<table>
<thead>
<tr>
<th>Scattering Angle</th>
<th>0.355 $\mu m$</th>
<th>0.532 $\mu m$</th>
<th>1.064 $\mu m$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>178°</td>
<td>0.21 ± 0.02</td>
<td>0.16 ± 0.02</td>
<td>0.09 ± 0.02</td>
<td>Mamouri and Ansmann (2015)</td>
</tr>
<tr>
<td>180 ± 0.2°</td>
<td>0.375 ± 0.015</td>
<td>0.350 ± 0.015</td>
<td>-</td>
<td>Miffre et al. (2016)</td>
</tr>
<tr>
<td>180 ± 0.2°</td>
<td>0.355 ± 0.015</td>
<td>0.305 ± 0.015</td>
<td>-</td>
<td>Miffre et al. (2016)</td>
</tr>
</tbody>
</table>
Impact of Real Part of the Refractive Index
Impact of the Imaginary Part of the Refractive Index
Small sizes

$r_m = 0.12\, \mu m$

$\sigma = 2.2$
Large sizes

\[ r_m = 1.50 \mu m \]
\[ \sigma = 2.2 \]
Fine mode + Coarse mode

Depolarization Ratio $\delta_{355}$

$r_f = 0.1 \mu m$
$\ln \sigma_f = 0.4$
$r_c = 1.0 \mu m$
$\ln \sigma_c = 0.6$
$N_c / N_f = 0.01$

Lidar Ratio $S_{355}$
Dust-soot Mixtures

(e.g., Sorenson, 2001)
Noncontact mixing is closer to observations.
Sea-salt optical modeling
Schematic of Sea-spray Aerosol Production

Dry sodium choride

(a) Image of dry sodium chloride crystals with a scale of 2 μm. (b) Image of dry sodium chloride crystals at a higher magnification, showing a cube-like structure with a scale of 400 nm. 

(c), (d), (e), (f) Models of dry sodium chloride crystals showing different shapes and sizes.
Aspect Ratio and Roundness Effect

Sakai et al. (2010)

NaCl

\[ r_m = 0.12 \mu m \]

\[ \sigma = 1.9 \]

Depolarization Ratio (~179°)

- a/c = 0.7
- a/c = 0.8
- a/c = 0.9
- a/c = 1.0

0.084 (perfect cube)
Wet Sea Salt Aerosols

X-Ray phasing technique (Zeng et al. 2012)
Depolarization VS Relative Humidity

Data from Toshiyuki Murayama
Effect of Particle Inhomogeneity (RH=50~80% inhomogeneous)

Global mean: 0.033/0.408 = 8%

Summary

• Significant progress has been made on computing the optical properties of non-spherical particles. Extensive computations are now allowable with reasonable computational resources and computational time.

• Comparison with measurement and observations show that super-ellipsoidal space is quite promising with applications in remote sensing and radiative transfer modeling.
The graph shows the depolarization ratio as a function of the effective radius (\(\mu m\)) and mean radius (\(\mu m\)).

- **Top Graph:**
  - Depolarization Ratio (%)
  - Effective Radius (\(\mu m\))
  - Three plotted curves:
    - Purple curve labeled as 355nm
    - Green curve labeled as 532nm
  - m = 1.53 \pm 0.008

- **Bottom Graph:**
  - Depolarization Ratio (%)
  - Mean Radius (\(\mu m\))
  - Two plotted curves:
    - Purple curve labeled as 355nm
    - Green curve labeled as 532nm
  - \(\sigma = 2.2\)
Tang et al., Optics Express, (2019)
Polarization Simulations
Dry sea salt

\[ \lambda = 550nm \]
\[ m = 1.50 + i1.00 \times 10^{-8} \]

sphere
\[ a/c = 1.0, n = 1.0 \]

superspheroid
\[ a/c = 0.9, n = 0.35 \]
Wet sea salt

\[ \lambda = 550\text{nm} \]

homogeneous sphere
\[ m_{\text{eff}} = 1.3845 + i4.70 \times 10^{-9} \]

inhomogeneous sphere
\[ m_{\text{core}} = 1.500 + i1.00 \times 10^{-8} \]
\[ m_{\text{layer}} = 1.3359 + i2.46 \times 10^{-9} \]