Improving coupled model solution mathematical consistency through data assimilation.

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Coupled Data Assimilation: an opportunity

OA coupling is a complex matter with many sources of uncertainties

- time/space non-conformity
- interfaces may actually not be represented by any component
- multi physics with different characteristics.
- highly parameterised interface (Bulk formulae)
- coupling methods
- ...

Some of these uncertainties are unavoidable, some others are linked to the way we implement things.
Coupled DA is an opportunity to account for or reduce them
Coupled modelling systems

\[ L_a u_a = f_a \]

\[ L_o u_o = f_o \]

- \( K_m \) parameterization in the boundary layers
- \( F_{oa} = (\tau, Q_{net}) \) is the interface flux

At the Interface \( \Gamma \):

\[ G_o u_o = G_a u_a \]
\[ F_a u_a = F_o u_o \]
**Coupling methods**

**Usual approaches**

- **Synchronous method.**
  - Aliasing errors
  - Synchronicity issues
  - Physics-dynamics inconsistency error

**Possible solutions**

- **Monolithic approach**
- **Iterative method to solve the coupling problem**

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**Figures**

- **SYNCHRONOUS**
  - Synchronous method.
  - Aliasing errors
  - Synchronicity issues
  - Physics-dynamics inconsistency error

- **ASYNCHRONOUS**
  - Asynchronous method:
    - Does not solve the original problem
Coupling methods
Schwarz Waveform Relaxation (AKA Global in time Schwarz method)

\[ \Omega_{\text{atm}} \]

\[ \Omega_{\text{oce}} \]

\[ t_{i-1} \quad i-1 \quad t_i \quad i \quad t_{i+1} \quad \text{time} \]

\[ F_{\text{oa}}(\langle U_o \rangle_i^{k-1}, U_a_i^{k-1}) \]

Considering:
- \( u_0 \in H^1(\Omega_a \cup \Omega_o) \) the initial condition
- \( k \) the iteration number
- \( u_a^0(0, t) \) the first-guess

The SWR algorithm reads:

\[
\begin{align*}
\mathcal{L}_o u_o^k &= f_o & \text{on } \Omega_o \times T_i \\
u_o^k(z, 0) &= u_0(z) & z \in \Omega_o \\
\mathcal{G}_o u_o^k &= \mathcal{G}_a u_a^k & \text{on } \Gamma \times T_i \\
F_a u_a^k &= F_o u_o^{k-1} & \text{on } \Gamma \times T_i
\end{align*}
\]

where \( T_i = [t_i; t_{i+1}] \)

- At convergence, it provides a flux consistent solution: \( F_a u_a = F_o u_o \) and \( \mathcal{G}_o u_o = \mathcal{G}_a u_a \) on \( \Gamma \times T_i \)
Coupling methods
Why does it matter

Hurricane Erica’s trajectory and ensemble spread
18 members of WRF/ROMS, generated through perturbations of initial conditions and coupling frequency (lemarié et al. 2014))
Coupling Methods
Usual coupling vs Schwarz methods

Main drawbacks:
- This is an iterative method
- Convergence speed greatly depends on $F_d$, $G_d$ and $u_0^0(0, t)$ ($d = a, o$)

Advantages:
- This is a non-intrusive coupling method
- At convergence, it provides a strongly coupled solution

Starting point of this ERA-CLIM2 task, in the framework of a variational system
- Can we improve the boundary conditions to accelerate the SWR convergence?
- Take benefit of the minimisation iterations for the SWR ones
Fully Iterative Method (FIM)

\( x_0 = u_0(z), \ z \in \Omega = \Omega_a \cup \Omega_o \) is the controlled state vector

\( x^{cvg} = (u_a^{k_{cvg}}, u_o^{k_{cvg}})^T \) is the converge solution of the SWR algorithm: \( k_{cvg} \) iterations

The first-guess \( u_a^0 \) in the SWR algorithm is updated after each minimisation iteration

\[
J_{FIM}(x_0) = J^b(x_0) + \int_{t_i}^{t_{i+1}} \left\langle y - H(x^{cvg}), R^{-1}(y - H(x^{cvg})) \right\rangle \Omega \ dt
\]

- The solution provided is strongly fully insanely coupled
- It requires the adjoint of the coupling
- It possibly requires a large number of Schwarz iterations
Truncated iterative method (TIM)

- \( x_0 = (u_0(z), u_0^0(0, t))^T, \quad z \in \Omega \setminus \Gamma \)
- The Schwarz iterations are truncated at \( k_{\text{max}} < k_{\text{cvg}} \) iterations
- \( x_{\text{max}} = (u_a^{k_{\text{max}}}, u_o^{k_{\text{max}}})^T \)
- Extended cost function:

\[
J^s = \alpha \mathcal{F} \| \mathcal{F}_a u_a^{k_{\text{max}}}(0, t) - \mathcal{F}_o u_o^{k_{\text{max}}}(0, t) \|^2_{T_i} + \alpha \mathcal{G} \| \mathcal{G}_a u_a^{k_{\text{max}}}(0, t) - \mathcal{G}_o u_o^{k_{\text{max}}}(0, t) \|^2_{T_i}
\]

with \( \| a \|^2_\Sigma = \langle a, a \rangle_\Sigma \)

\[
J_{\text{TIM}}(x_0) = J^b(x_0) + \int_{t_i}^{t_{i+1}} \left\langle y - H(x_{\text{max}}), R^{-1}(y - H(x_{\text{max}})) \right\rangle_\Omega \, dt + \boxed{J^s}
\]

- The solution provided is quasi-strongly coupled
- It requires the adjoint of the coupling
- It requires fewer number of Schwarz iterations than the FIM
Weakly Interfaced Models (WIM)

- \( x_0 = (x_{0,a}, x_{0,o})^T \) with \( x_{0,d} = (u_0|_{z \in \Omega_d}, u_d^0(0, t)) \)
- The direct coupling between both models is suppressed
- Models are coupled during the assimilation process

\[
J_{WIM}(x_0) = \left\{ \sum_{d=a,o} (J^b(x_{0,d}) + J^o(x_{0,d})) \right\} + J^s
\]

- The solution provided is weakly coupled (as coupling is a weak constraint)
- It requires only the adjoints of the uncoupled models
- There is no coupling iterations
### Considered schemes - Summary

<table>
<thead>
<tr>
<th>Algo</th>
<th>Control vector</th>
<th># of coupling iterations</th>
<th>extended cost function</th>
<th>Adjoint of the coupling</th>
<th>Coupling</th>
</tr>
</thead>
<tbody>
<tr>
<td>FIM</td>
<td>( u_0(z) )</td>
<td>( k_{cvg} )</td>
<td>no</td>
<td>yes</td>
<td>strong</td>
</tr>
<tr>
<td>TIM</td>
<td>( (u_0(z), u_0^o) )^T</td>
<td>( k_{max} )</td>
<td>possibly</td>
<td>yes</td>
<td>~strong</td>
</tr>
<tr>
<td>WIM</td>
<td>( (u_0(z), u_0^o, u_0^o) )^T</td>
<td>1</td>
<td>yes</td>
<td>no</td>
<td>weak</td>
</tr>
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**Table:** Overview of the properties of the coupled variational DA methods described.
Application to a 1D diffusion problem
Our simple coupled system: two coupled 1D diffusion equations

Previous algorithms (in incremental formulation) are applied on a **1D linear diffusion problem**, using the OOPS framework.
Let us consider \((d = a, o)\):

- \(\mathcal{L}_d = \partial_t + \nu_d \partial_z^2\)
- \(\nu_a \neq \nu_o\) the diffusion coefficients
- \(\mathcal{G}_d = \nu_d \partial_z\) and \(\mathcal{F}_d = \text{Id}\) the interface operators on \(\Gamma\)
- \(f_d\) the second member such that the analytical solution is

\[
u_d^*(z, t) = \frac{u_0}{4} e^{-\frac{|z|}{\alpha_d}} \left\{ 3 + \cos^2 \left( \frac{3\pi t}{\tau} \right) \right\} \text{ on } \Omega_d \times T_i
\]
Application to a 1D diffusion problem

Our simple coupled system: two coupled 1D diffusion equations - results

<table>
<thead>
<tr>
<th>Algo</th>
<th>$\alpha_F$</th>
<th>$\alpha_G$</th>
<th>$k_{max}$</th>
<th># of minimisation iterations</th>
<th># of models runs</th>
<th>Interface imbalance indicator</th>
<th>RMSE in °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>FIM</td>
<td>-</td>
<td>-</td>
<td>$k_{cvg}$</td>
<td>14</td>
<td>1501</td>
<td>$5 \cdot 10^{-12}$</td>
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<tr>
<td>TIM</td>
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<td>1509</td>
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<tr>
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<tr>
<td><strong>TIM</strong></td>
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<td>-</td>
<td><strong>1</strong></td>
<td><strong>97</strong></td>
<td><strong>197</strong></td>
<td><strong>1 \cdot 10^{-6}</strong></td>
<td><strong>0.217</strong></td>
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<td>835</td>
<td>$1.4 \cdot 10^{-7}$</td>
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<td>0</td>
<td>1</td>
<td>24</td>
<td>51</td>
<td>2633.</td>
<td>8.495</td>
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### Application to a 1D diffusion problem

Our simple coupled system: two coupled 1D diffusion equations - fixed number of iteration

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<td>5.10^{-5}</td>
<td>0.330</td>
</tr>
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<td>WIM</td>
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<td>40</td>
<td>1</td>
<td>24</td>
<td>51</td>
<td>7.10^{-1}</td>
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<tr>
<td>Uncoupled</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>24</td>
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<td>CERA-like-2</td>
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<td>–</td>
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<tr>
<td>CERA-like-4</td>
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<td>–</td>
<td>2</td>
<td>$4 \times 5$</td>
<td>54</td>
<td>$7 \times 10^{-1}$</td>
<td>0.3</td>
</tr>
</tbody>
</table>
Final comments

It is difficult to draw a clear conclusion from such a simplistic testcase but

- The way models are coupled should not be overlooked
- but controlling and/or penalising the interface mismatch could be a step toward stronger coupling

Before the end of the project

- more in depth theoretical study on convergence
- Apply these algorithms to a more realistic coupled SCM (Ocean/ABL, currently being implemented within OOPS)
- look into optimized interface conditions for SWR

In parallel:

- extend this work to ensemble smoother