Applying the Laplace Transform Integration Scheme in OpenIFS

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Aim of Research

Implement a new time integration scheme into a state of the art weather forecast model.

The new scheme (LTIS) is based on the Laplace Transform.

Ideally any new scheme would be more accurate and computationally competitive with operational 2TL SISL scheme.

Work with OpenIFS, the open version of the ECMWF's IFS.

Content of Presentation

Firstly I will discuss the basics of the LTIS.

Then I will cover the completed work on the Eulerian SW model.

Currently working on the Semi-Lagrangian SW model.

Finally I will outline the future plans for our research.

History of LTIS

LT filtering properties used in initialisation by Lynch (MWR 1985), (QJMS 1985).

First used for numerical integration by Van Isacker and Struylaert (1985), Lynch (1986) and Lynch (MWR 1991).

Further work by Clancy and Lynch, (QJMS 2011) & (QJMS 2014).

For a function $f(t)$ with $t \geq 0$, its Laplace Transform (LT) is

$$\hat{f}(s) = \mathcal{L}f(t) = \int_0^\infty e^{-st} f(t) dt$$

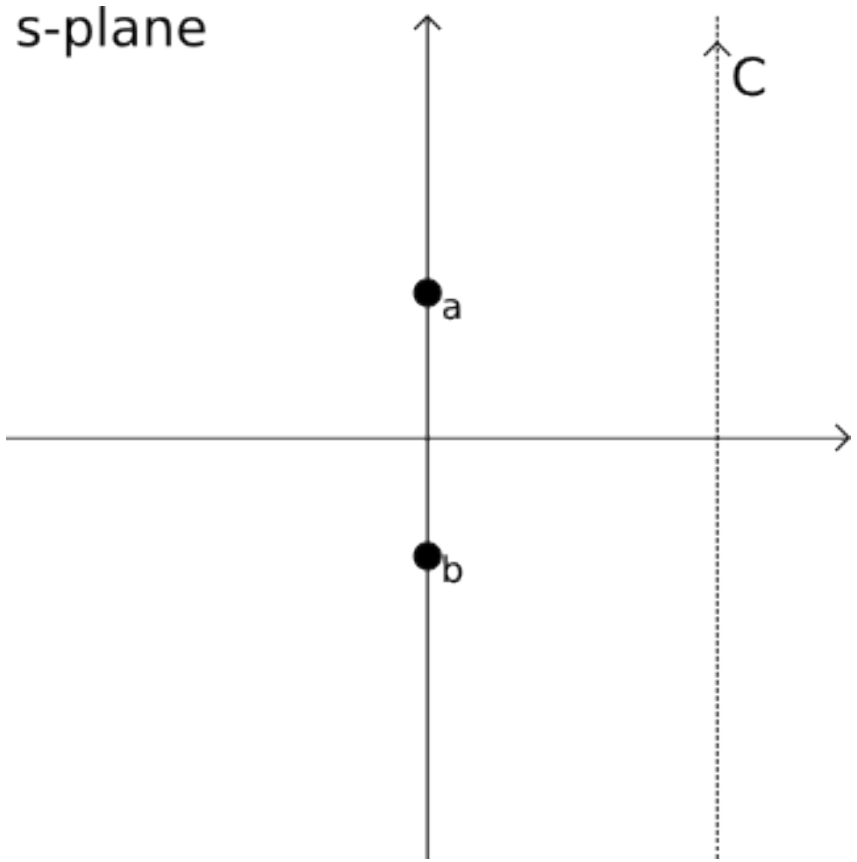
where $s \in \mathbb{C}$.

Recover original function via Inverse Laplace Transform

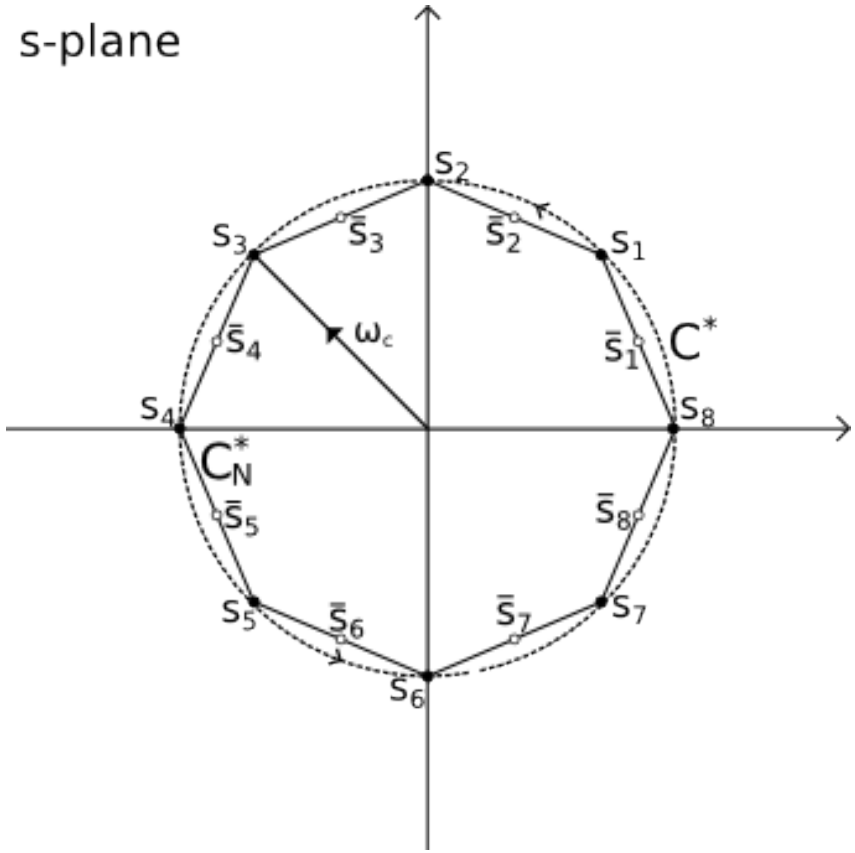
$$f(t) = \mathcal{L}^{-1}\hat{f}(s) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{st} \hat{f}(s) ds.$$

Basics of LTIS

Contour $\mathcal{C}$ is line parallel to imaginary axis in the s-plane, to the right of the singularities (simple poles) of $\hat{f}(s)$, closed by a semicircle at infinity.



Choose a contour $\mathcal{C}^*$, defined by cut-off frequency ω_c, so that higher frequency components will lie outside and be filtered.



The modified LT inversion can be performed either,

Analytically:

$$\mathcal{L}^* \left\{ \hat{f}(s) \right\} = \frac{1}{2\pi i} \int_{\mathcal{C}^*} e^{st} \hat{f}(s) ds.$$

Numerically:

$$\mathcal{L}_N^* \left\{ \hat{f}(s) \right\} = \frac{1}{2\pi i} \sum_{i=1}^{N} e^{s_n t} \hat{f}(s_n) \Delta s_n.$$

The choice depends on the form of the equations in s-space.

Consider a general system for a state $\mathbf{X}(t)$ governed by equation,

$$\frac{\partial \mathbf{X}}{\partial t} + \mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = 0,$$

where $\mathbf{L}$ is a linear operator and $\mathbf{N}$ is a nonlinear vector function. Taking LT of the equation gives

$$s\hat{\mathbf{X}} - \mathbf{X}_0 + \mathbf{L}\hat{\mathbf{X}} + \hat{\mathbf{N}} = 0,$$

where $\mathbf{X}_0$ is the system at $t = 0$. Can approximate $\mathbf{N}(\mathbf{X})$ by constant vector $\mathbf{N}_0 = \mathbf{N}(\mathbf{X}_0)$, then have

$$\hat{\mathbf{X}} = (s\mathbf{I} + \mathbf{L})^{-1} \left[\mathbf{X}_0 - \mathbf{N}_0/s\right].$$

Consider time interval $[n\Delta t, (n+1)\Delta t]$, apply modified inverse LT at time Δt, giving numerical integration scheme

$$\hat{\mathbf{X}} = (s\mathbf{I} + \mathbf{L})^{-1} [\mathbf{X}_n - \mathbf{N}_n/s]$$
$$\mathbf{X}^{n+1} = \mathcal{L}^* \left\{ \hat{\mathbf{X}} \right\}_{\Delta t}$$

Analytic modified inversion on time interval $[n\Delta t, (n+1)\Delta t]$,

$$\mathcal{L}^* \left\{ \hat{\mathbf{X}}(s) \right\} = \frac{1}{2\pi i} \int_{\mathcal{C}^*} e^{s\Delta t} \hat{\mathbf{X}}(s) ds.$$

Simple Filtering Example

Consider the function,

$$f(t) = ae^{i\omega t} + Ae^{i\Omega t},$$

where $|\omega| \ll |\Omega|$. Take LT of $f(t)$,

$$\hat{f}(s) = \frac{a}{s - i\omega} + \frac{A}{s - i\Omega}.$$

Poles at $s = i\omega$ & $s = i\Omega$, choose contour $\mathcal{C}^*$ so that $\omega < \omega_c < \Omega$. Inverting gives filtered function,

$$f^*(t) = ae^{i\omega t}.$$

Simple ODE Example

Consider the ODE (Oscillation Equation),

$$\frac{\partial u}{\partial t} = i\omega u.$$

Take LT on time interval $[n\Delta t, (n+1)\Delta t]$,

$$s\hat{u} - u^n = i\omega\hat{u},$$
$$\Rightarrow \hat{u} = \frac{u^n}{s - i\omega}.$$

Inverting gives exact result,

$$u^{n+1} = u^n e^{i\omega\Delta t}.$$

Benefits of LTIS

LTIS has two beneficial properties.

- Filter unwanted high frequency components.
- Exact result for simple linear terms.

Requires system of PDEs to be reduced to system of ODEs, spectral transform method simplifies this greatly.

OpenIFS

Managed by Glenn Carver (Numerical Aspects) and Filip Vaña (Physical Aspects).

Includes a Shallow Water Model running a 3TL EUSI scheme and the 2TL SISL scheme.

Changes kept to a minimum.

All results from runs using T255.

LTIS in Eulerian Shallow Water Model

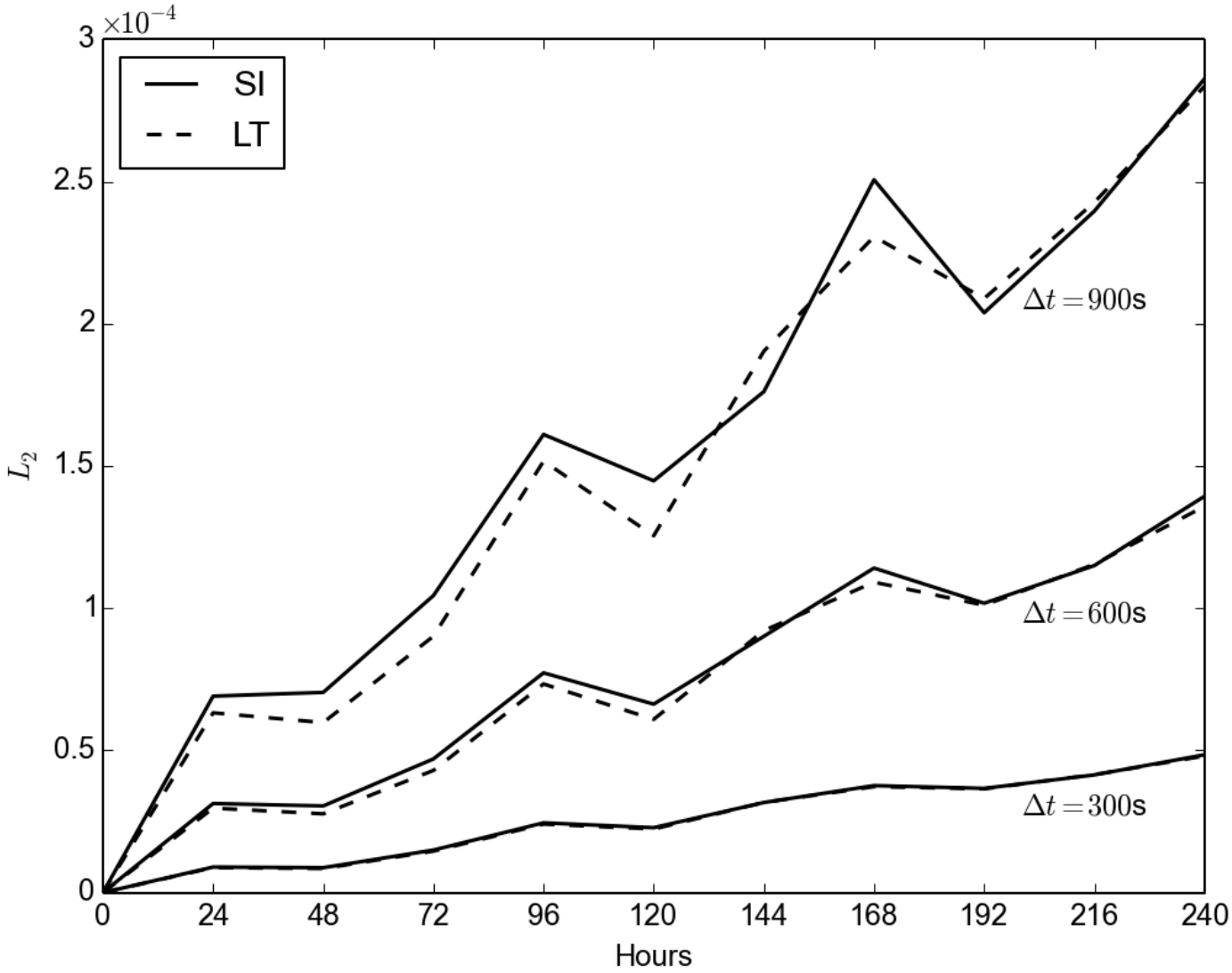
Eulerian equations take form,

$$\frac{\partial \mathbf{X}}{\partial t} + \mathbf{L} + \mathbf{N} = 0,$$
$$\Rightarrow \frac{\partial \mathbf{X}}{\partial t} + \mathbf{L} = -\mathbf{N}^n = \mathbf{R}$$

Applying spectral transform gives system of ODEs for spectral coefficients, which can be solved by the LTIS.

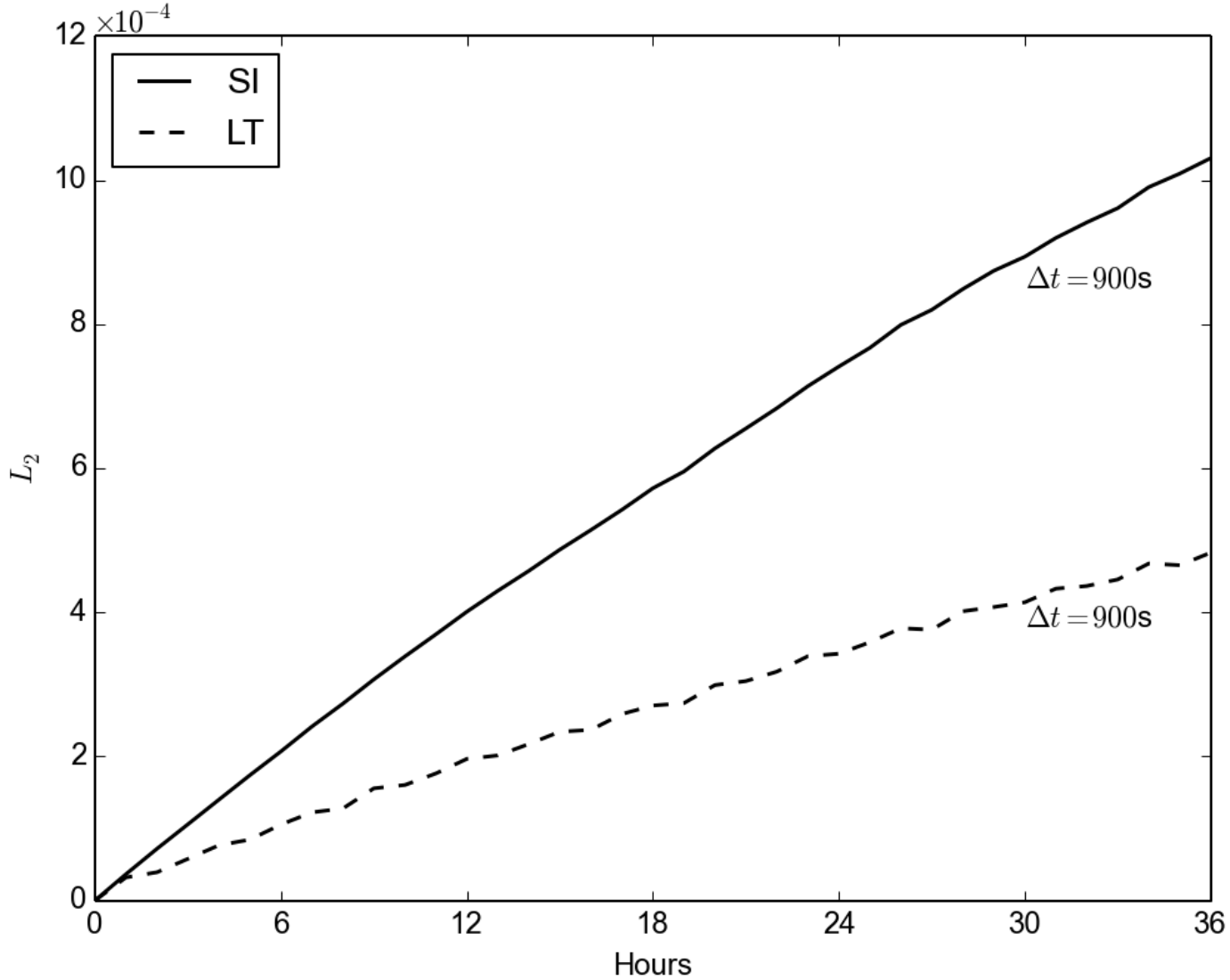
LTIS in Eulerian Shallow Water Model

L_2 error for geopotential, for *Läuter (JCP, 2005)* analytic unsteady solid body rotation solution. Using $\tau_c = 1\text{hr}$, where $\tau_c = 2\pi/\omega_c$.



LTIS in Eulerian Shallow Water Model

L_2 error for geopotential, approx Kelvin Wave with wave number m=6, compared to reference solution (EUSI $\Delta t = 300$s).



Semi-Lagrangian Schemes

IFS uses a 2TL Semi-Implicit Semi-Lagrangian (SISL) scheme.

Consider an equation:

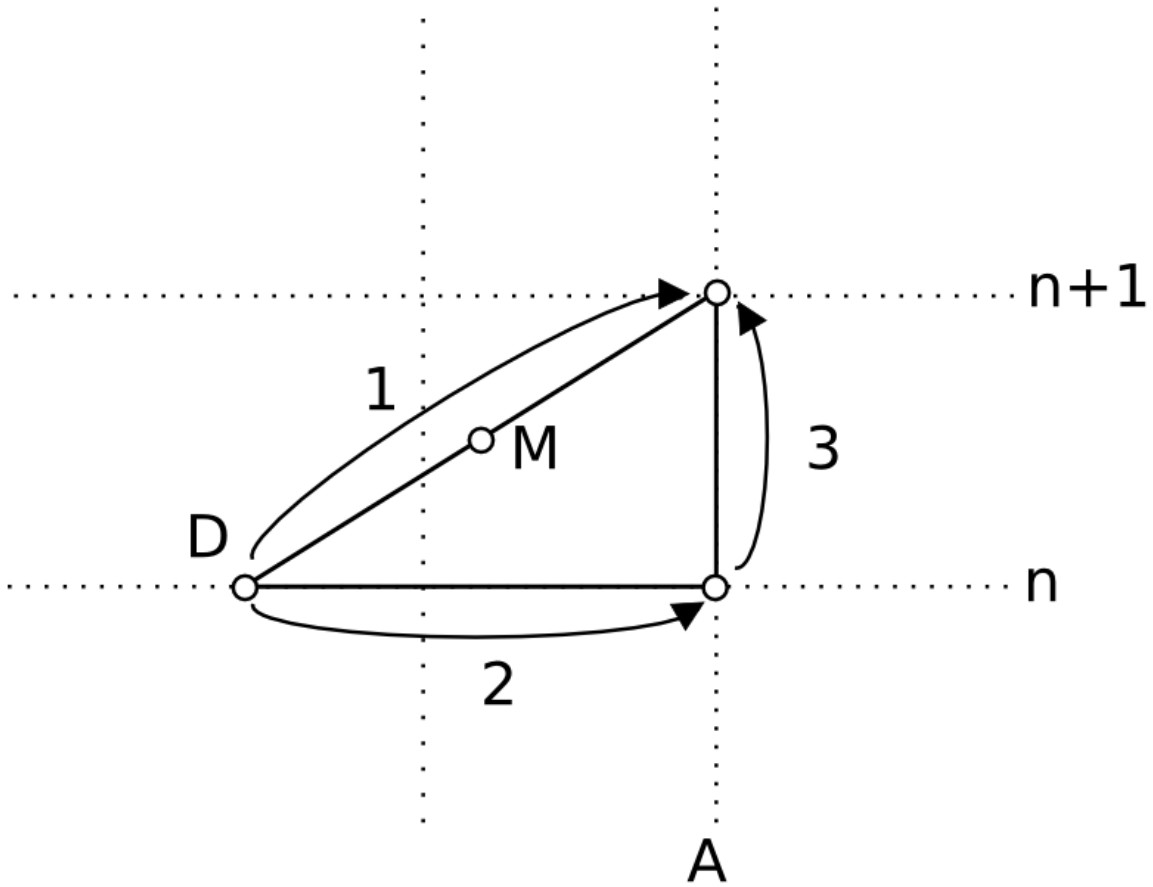
$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + \mathbf{L} + \mathbf{N} = 0$$

The 2TL SISL scheme approximates this equation as:

$$\frac{\mathbf{X}_A^{n+1} - \mathbf{X}_D^n}{\Delta t} + \frac{\mathbf{L}_A^{n+1} + \mathbf{L}_D^n}{2} + \frac{\mathbf{N}_A^{n+1/2} + \mathbf{N}_D^{n+1/2}}{2} = 0$$

2TL SISL:

$$\frac{\mathbf{X}_A^{n+1} - \mathbf{X}_D^n}{\Delta t} + \frac{\mathbf{L}_A^{n+1} + \mathbf{L}_D^n}{2} + \frac{\mathbf{N}_A^{n+1/2} + \mathbf{N}_D^{n+1/2}}{2} = 0$$



LTIS in Semi-Lagrangian Shallow Water Model

In previous work on SL, LT was applied in grid space along 1, this has issues and complications:

- Commutation of operators ($\mathcal{L}$ and spatial derivatives).
- Extra spectral transforms.
- Work (and write code) in $\mathbb{C}$ s-space.
- Didn't use analytic inversion.

Stably Advected Laplace Transform (SALT) scheme (*Lynch (1991)*), splits total derivative into spatial and temporal parts.

Extended to linear terms, SALT2 approach.

2TL SISL scheme,

$$\frac{\mathbf{X}_A^{n+1} - \mathbf{X}_D^n}{\Delta t} + \frac{\mathbf{L}_A^{n+1} + \mathbf{L}_D^n}{2} + \frac{\mathbf{N}_A^{n+1/2} + \mathbf{N}_D^{n+1/2}}{2} = 0$$

Write as,

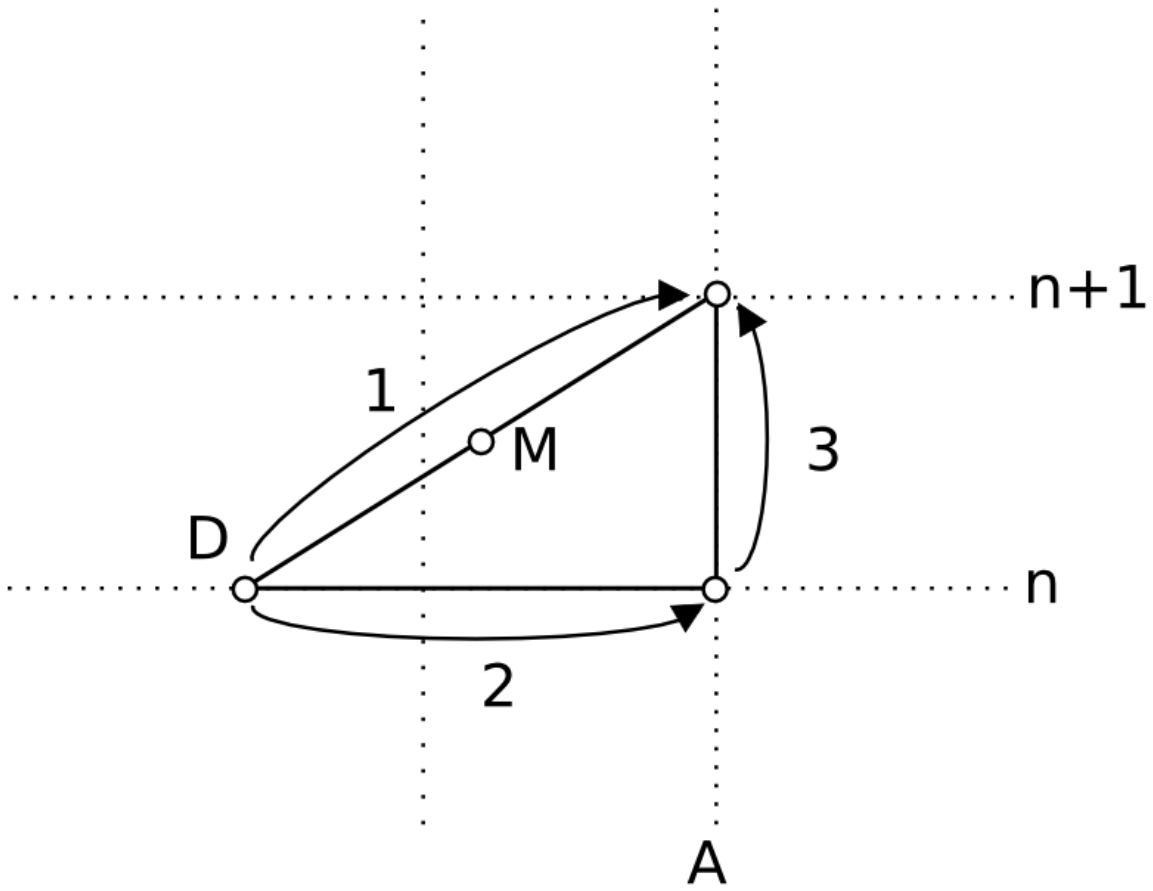
$$\frac{\mathbf{X}_A^{n+1} - \mathbf{X}_A^n}{\Delta t} + \frac{\mathbf{X}_A^n - \mathbf{X}_D^n}{\Delta t} + \frac{\mathbf{L}_A^{n+1} + \mathbf{L}_A^n}{2} + \frac{\mathbf{L}_D^n - \mathbf{L}_A^n}{2} + \frac{\mathbf{N}_A^{n+1/2} + \mathbf{N}_D^{n+1/2}}{2} = 0$$

$$\frac{\partial \mathbf{X}}{\partial t} + \frac{\mathbf{X}_A^n - \mathbf{X}_D^n}{\Delta t} + \mathbf{L}_A + \frac{\mathbf{L}_D^n - \mathbf{L}_A^n}{2} + \frac{\mathbf{N}_A^{n+1/2} + \mathbf{N}_D^{n+1/2}}{2} = 0$$

This is the SALT2 approach.

SALT2 form of equations,

$$\frac{\partial \mathbf{X}}{\partial t} + \frac{\mathbf{X}_A^n - \mathbf{X}_D^n}{\Delta t} + \mathbf{L}_A + \frac{\mathbf{L}_D^n - \mathbf{L}_A^n}{2} + \frac{\mathbf{N}_A^{n+1/2} + \mathbf{N}_D^{n+1/2}}{2} = 0$$



LTIS in Semi-Lagrangian Shallow Water Model

At end of grid space calculations we have,

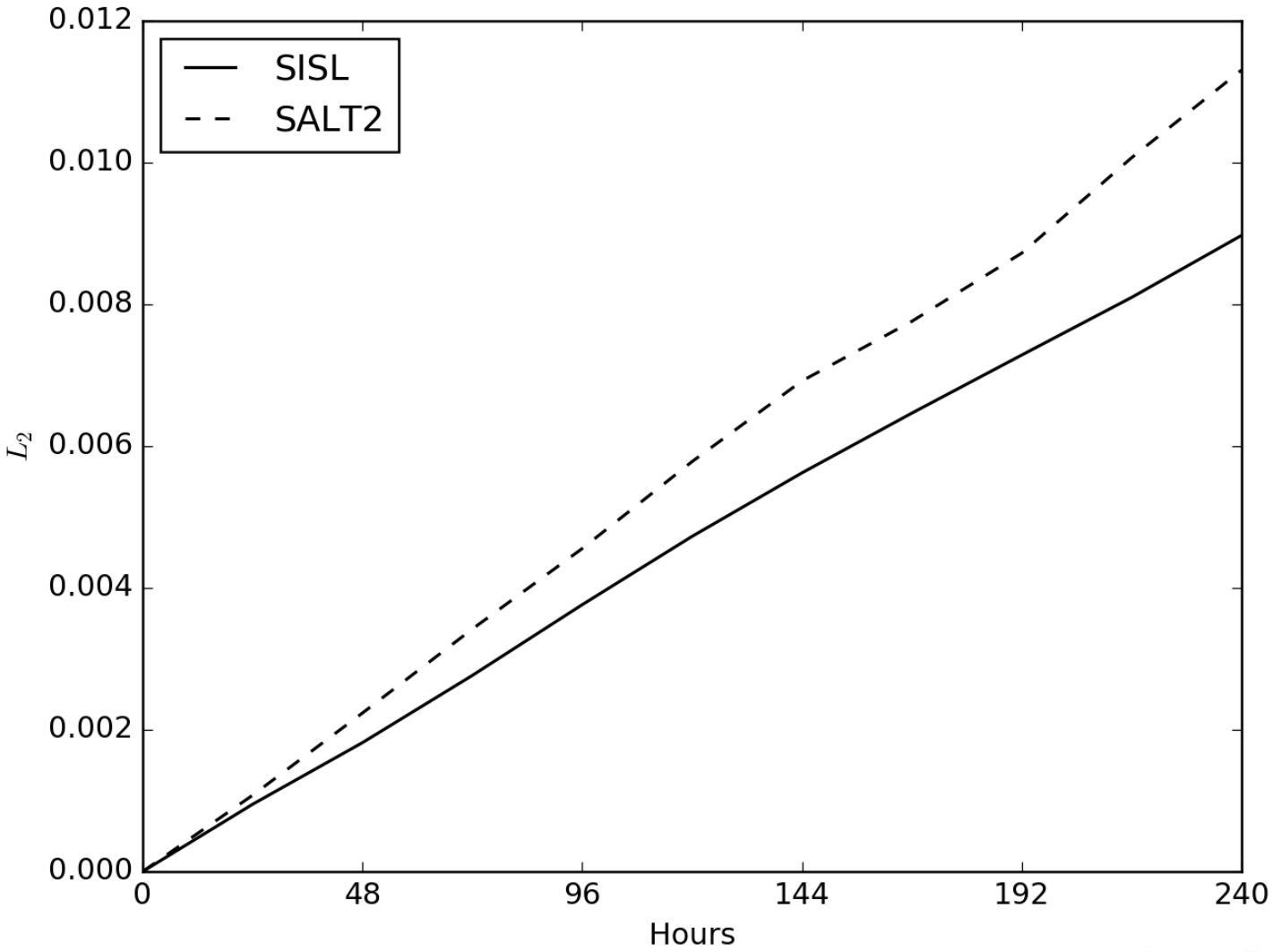
$$\frac{\partial \mathbf{X}}{\partial t} + \mathbf{L}_A = \mathbf{R}$$

R includes advection terms, spatial part of linear terms, Coriolis and non-linear terms.

Transforming to spectral space, gives the same system of ODEs as the Eulerian scheme.

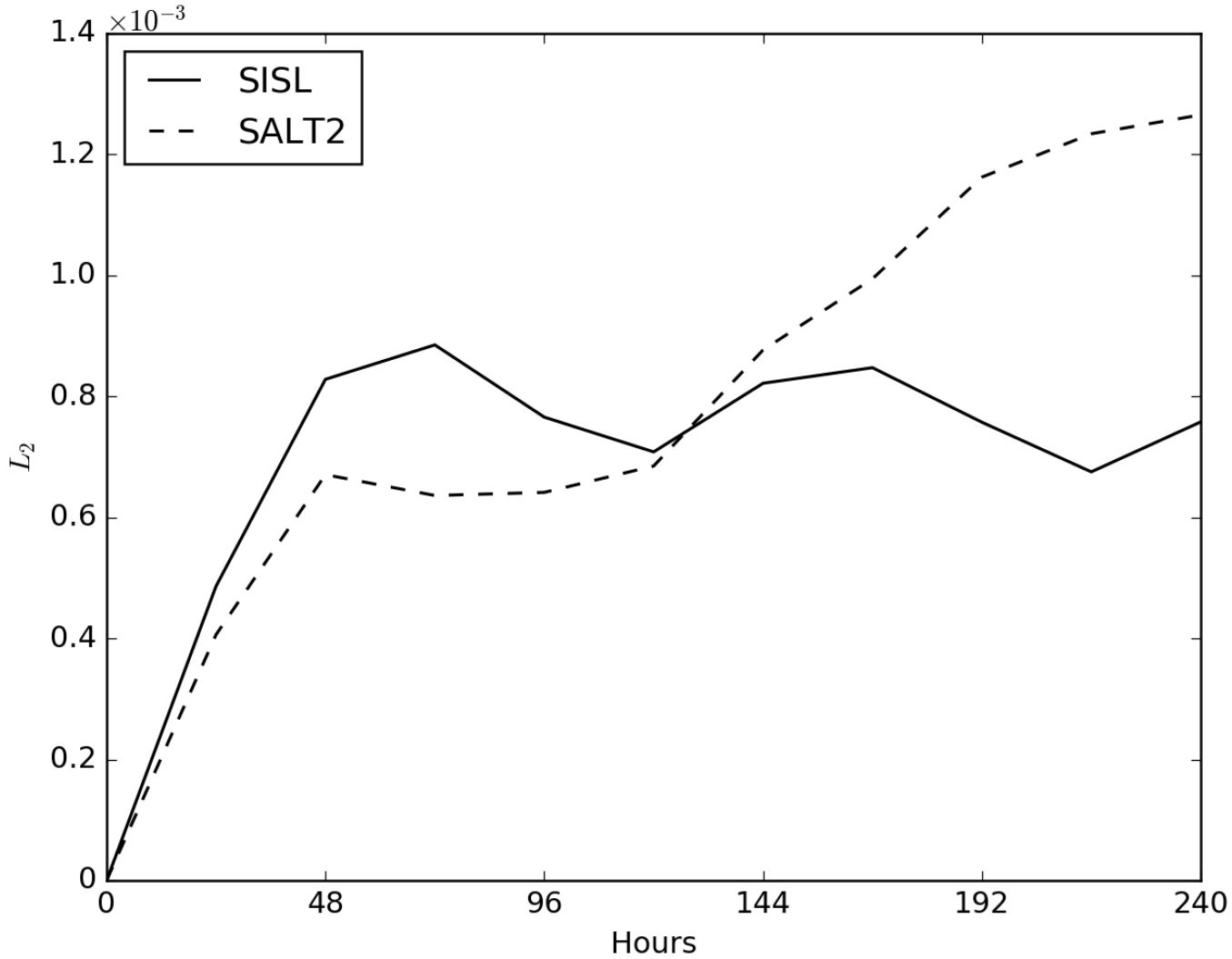
LTIS in Semi-Lagrangian Shallow Water Model

L_2 error for geopotential, for *Läuter (2005)* analytic unsteady solid body rotation solution. ($\Delta t = 1800$s and $\tau_c = 6$hrs.)



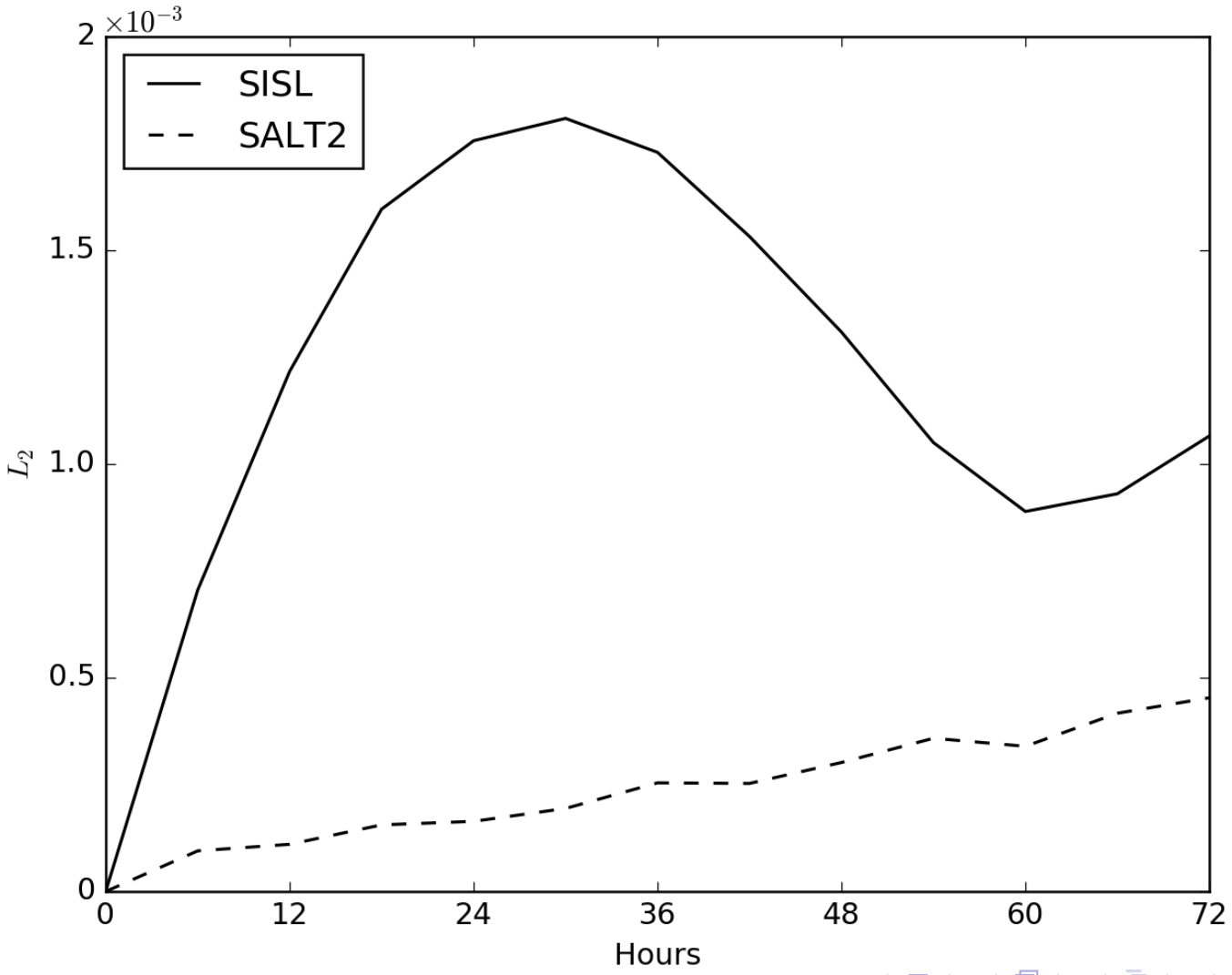
LTIS in Semi-Lagrangian Shallow Water Model

L_2 error for geopotential, for Mountain Case (from Williamson Test Cases). ($\Delta t = 1800$s and $\tau_c = 6$hrs.) Reference EUSI $\Delta t = 300$s.



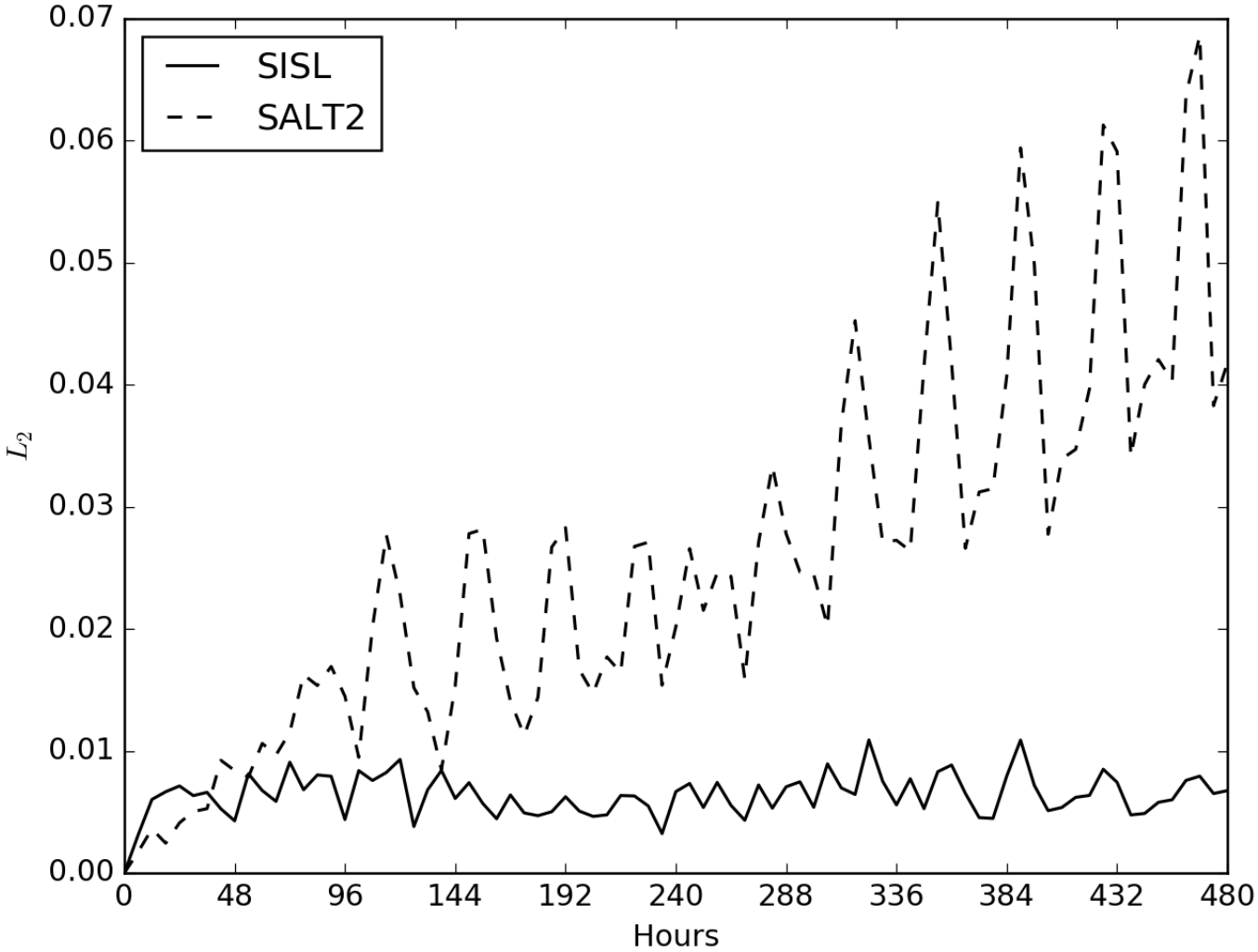
LTIS in Semi-Lagrangian Shallow Water Model

L_2 error for geopotential, for approximate m=6 Kelvin Wave. ($\Delta t = 1800$s and $\tau_c = 3$hrs.) Reference EUSI $\Delta t = 300$s.



LTIS in Semi-Lagrangian Shallow Water Model

L_2 error for geopotential, for Rossby-Haurwitz Wave Case. ($\Delta t = 1800$s and $\tau_c = 6$hrs.) Reference EUSI $\Delta t = 300$s.



Potential Issues

Investigating whether this is due to an issue in the implementation or the scheme itself.

Another issue is use of large τ_c (Ideally should be smaller).

Further testing of SALT2 is planned, in particular running a real data case.

Conclusions and Future Work

- Results are not yet as good as hoped.
- Computationally competitive (no overhead compared to 2TL SISL).
- Issues with accuracy will (hopefully) soon be fixed.

Next objective is implementation into the full baroclinic model of OpenIFS.

This will require a thorough examination of current 2TL SISL scheme in the full model and choice of equation set.

End

Thanks for listening.