Applying the Laplace Transform Integration Scheme in OpenIFS

Eoghan Harney

Supervisor: Prof Peter Lynch

University College Dublin

October 4th, 2016
Aim of Research

Implement a new time integration scheme into a state of the art weather forecast model.

The new scheme (LTIS) is based on the Laplace Transform.

Ideally any new scheme would be more accurate and computationally competitive with operational 2TL SISL scheme.

Work with OpenIFS, the open version of the ECMWF’s IFS.
Firstly I will discuss the basics of the LTIS.

Then I will cover the completed work on the Eulerian SW model.

Currently working on the Semi-Lagrangian SW model.

Finally I will outline the future plans for our research.
History of LTIS

LT filtering properties used in initialisation by Lynch (MWR 1985), (QJMS 1985).


Basics of LTIS

For a function $f(t)$ with $t \geq 0$, its Laplace Transform (LT) is

$$\hat{f}(s) = \mathcal{L}f(t) = \int_0^\infty e^{-st}f(t)dt$$

where $s \in \mathbb{C}$.

Recover original function via Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\hat{f}(s) = \frac{1}{2\pi i} \int_C e^{st}\hat{f}(s)ds.$$
Contour $\mathcal{C}$ is line parallel to imaginary axis in the $s$-plane, to the right of the singularities (simple poles) of $\hat{f}(s)$, closed by a semicircle at infinity.
Choose a contour $C^*$, defined by cut-off frequency $\omega_c$, so that higher frequency components will lie outside and be filtered.
The modified LT inversion can be performed either,

Analytically:

\[ \mathcal{L}^* \left\{ \hat{f}(s) \right\} = \frac{1}{2\pi i} \int_{\mathcal{C}^*} e^{st} \hat{f}(s) ds. \]

Numerically:

\[ \mathcal{L}^*_N \left\{ \hat{f}(s) \right\} = \frac{1}{2\pi i} \sum_{i=1}^{N} e^{s_n t} \hat{f}(s_n) \Delta s_n. \]

The choice depends on the form of the equations in s-space.
Consider a general system for a state \( X(t) \) governed by equation,

\[
\frac{\partial X}{\partial t} + LX + N(X) = 0,
\]

where \( L \) is a linear operator and \( N \) is a nonlinear vector function. Taking LT of the equation gives

\[
s\hat{X} - X_0 + L\hat{X} + \hat{N} = 0,
\]

where \( X_0 \) is the system at \( t = 0 \). Can approximate \( N(X) \) by constant vector \( N_0 = N(X_0) \), then have

\[
\hat{X} = (sI + L)^{-1} [X_0 - N_0/s].
\]
Consider time interval \([n\Delta t, (n + 1)\Delta t]\), apply modified inverse LT at time \(\Delta t\), giving numerical integration scheme

\[
\hat{X} = (sI + L)^{-1} [X_n - N_n/s]
\]

\[
X^{n+1} = L^* \{\hat{X}\}_\Delta t
\]

Analytic modified inversion on time interval \([n\Delta t, (n + 1)\Delta t]\),

\[
L^* \{\hat{X}(s)\} = \frac{1}{2\pi i} \int_{C^*} e^{s\Delta t} \hat{X}(s) ds.
\]
Consider the function,

\[ f(t) = ae^{i\omega t} + Ae^{i\Omega t}, \]

where \(|\omega| \ll |\Omega|\). Take LT of \(f(t)\),

\[ \hat{f}(s) = \frac{a}{s - i\omega} + \frac{A}{s - i\Omega}. \]

Poles at \(s = i\omega\) \& \(s = i\Omega\), choose contour \(C^*\) so that \(\omega < \omega_c < \Omega\). Inverting gives filtered function,

\[ f^*(t) = ae^{i\omega t}. \]
Consider the ODE (Oscillation Equation),

\[ \frac{\partial u}{\partial t} = i\omega u. \]

Take LT on time interval \([n\Delta t, (n+1)\Delta t]\),

\[ s\hat{u} - u^n = i\omega \hat{u}, \]

\[ \Rightarrow \hat{u} = \frac{u^n}{s - i\omega}. \]

Inverting gives exact result,

\[ u^{n+1} = u^n e^{i\omega \Delta t}. \]
LTIS has two beneficial properties.

- Filter unwanted high frequency components.
- Exact result for simple linear terms.

Requires system of PDEs to be reduced to system of ODEs, spectral transform method simplifies this greatly.
Managed by Glenn Carver (Numerical Aspects) and Filip Vaña (Physical Aspects).

Includes a Shallow Water Model running a 3TL EUSI scheme and the 2TL SISL scheme.

Changes kept to a minimum.

All results from runs using T255.
Eulerian equations take form,

$$\frac{\partial \mathbf{X}}{\partial t} + \mathbf{L} + \mathbf{N} = 0,$$

$$\Rightarrow \frac{\partial \mathbf{X}}{\partial t} + \mathbf{L} = -\mathbf{N}^n = \mathbf{R}$$

Applying spectral transform gives system of ODEs for spectral coefficients, which can be solved by the LTIS.
$L_2$ error for geopotential, for Läuter (JCP, 2005) analytic unsteady solid body rotation solution. Using $\tau_c = 1\text{hr}$, where $\tau_c = \frac{2\pi}{\omega_c}$. 

![Graph showing $L_2$ error over time for different time steps $\Delta t$. The graph compares SI and LT methods.](image-url)
$L_2$ error for geopotential, approx Kelvin Wave with wave number $m=6$, compared to reference solution (EUSI $\Delta t = 300s$).
Semi-Lagrangian Schemes

IFS uses a 2TL Semi-Implicit Semi-Lagrangian (SISL) scheme.

Consider an equation:

\[
\frac{dX}{dt} + L + N = 0
\]

The 2TL SISL scheme approximates this equation as:

\[
\frac{X^{n+1}_A - X^n_D}{\Delta t} + \frac{L^{n+1}_A + L^n_D}{2} + \frac{N^{n+1/2}_A + N^{n+1/2}_D}{2} = 0
\]
2TL SISL:

\[
\frac{X_A^{n+1} - X_D^n}{\Delta t} + \frac{L_A^{n+1} + L_D^n}{2} + \frac{N_A^{n+1/2} + N_D^{n+1/2}}{2} = 0
\]
In previous work on SL, LT was applied in grid space along 1, this has issues and complications:

- Commutation of operators ($\mathcal{L}$ and spatial derivatives).
- Extra spectral transforms.
- Work (and write code) in $\mathbb{C}$ s-space.
- Didn’t use analytic inversion.

Stably Advected Laplace Transform (SALT) scheme (**Lynch (1991)**), splits total derivative into spatial and temporal parts.

Extended to linear terms, SALT2 approach.
2TL SISL scheme,

\[
\frac{X_A^{n+1} - X_D^n}{\Delta t} + \frac{L_A^{n+1} + L_D^n}{2} + \frac{N_A^{n+1/2} + N_D^{n+1/2}}{2} = 0
\]

Write as,

\[
\frac{X_A^{n+1} - X_A^n}{\Delta t} + \frac{X_A^n - X_D^n}{\Delta t} + \frac{L_A^{n+1} + L_D^n}{2} + \frac{L_D^n - L_A^n}{2} + \frac{N_A^{n+1/2} + N_D^{n+1/2}}{2} = 0
\]

\[
\frac{\partial X}{\partial t} + \frac{X_A^n - X_D^n}{\Delta t} + L_A + \frac{L_D^n - L_A^n}{2} + \frac{N_A^{n+1/2} + N_D^{n+1/2}}{2} = 0
\]

This is the SALT2 approach.
SALT2 form of equations,

\[
\frac{\partial \mathbf{X}}{\partial t} + \frac{\mathbf{X}_A^n - \mathbf{X}_D^n}{\Delta t} + L_A + \frac{L_D^n - L_A^n}{2} + \frac{N_{A}^{n+1/2} + N_{D}^{n+1/2}}{2} = 0
\]
At end of grid space calculations we have,

\[ \frac{\partial \mathbf{X}}{\partial t} + L_A = R \]

\( R \) includes advection terms, spatial part of linear terms, Coriolis and non-linear terms.

Transforming to spectral space, gives the same system of ODEs as the Eulerian scheme.
$L_2$ error for geopotential, for Läuter (2005) analytic unsteady solid body rotation solution. ($\Delta t = 1800\text{s and } \tau_c = 6\text{hrs.}$)
$L_2$ error for geopotential, for Mountain Case (from Williamson Test Cases). ($\Delta t = 1800s$ and $\tau_c = 6hrs.$) Reference EUSI $\Delta t = 300s$. 
$L_2$ error for geopotential, for approximate m=6 Kelvin Wave. ($\Delta t = 1800\text{s}$ and $\tau_c = 3\text{hrs.}$) Reference EUSI $\Delta t = 300\text{s}$. 
$L_2$ error for geopotential, for Rossby-Haurwitz Wave Case. ($\Delta t = 1800s$ and $\tau_c = 6hrs.$) Reference EUSI $\Delta t = 300s.$
Potential Issues

Investigating whether this is due to an issue in the implementation or the scheme itself.

Another issue is use of large $\tau_c$ (Ideally should be smaller).

Further testing of SALT2 is planned, in particular running a real data case.
Conclusions and Future Work

- Results are not yet as good as hoped.
- Computationally competitive (no overhead compared to 2TL SISL).
- Issues with accuracy will (hopefully) soon be fixed.

Next objective is implementation into the full baroclinic model of OpenIFS.

This will require a thorough examination of current 2TL SISL scheme in the full model and choice of equation set.
Thanks for listening.