AN UNSTRUCTURED MESH NFT APPROACH TO ALL-SCALE ATMOSPHERIC FLOWS

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Mass, entropy and momentum conservation laws:

\[
\frac{\partial G_0}{\partial t} + \nabla \cdot (G_0 \nu) = 0
\]

\[
\frac{\partial G_0 \theta'}{\partial t} + \nabla \cdot (G_0 \nu \theta') = -G_0 \left( \tilde{G}^T u \cdot \nabla \theta_a - H \right)
\]

\[
\frac{\partial G_0 u}{\partial t} + \nabla \cdot (G_0 \nu \otimes u) = -G_0 \left( \Theta \tilde{G} \nabla \varphi + g \gamma_B \frac{\theta'}{\theta_b} + f \times (u - \gamma_C u_a) - M' (u, u, \gamma_C) - D \right)
\]

\[\begin{align*}
\varphi &:= [c_p \theta_0 \pi', c_p \theta_0 \pi, c_p \theta_b \pi'] \\
\Theta &:= \left[ \frac{\theta}{\theta_0}, \frac{\theta}{\theta_0}, 1 \right] \\
\gamma_B &:= \left[ \frac{\theta_b(z)}{\theta_a(x)}, \frac{\theta_b(z)}{\theta_a(x)}, 1 \right] \\
\gamma_C &:= \left[ \frac{\theta_a(x)}{\theta_a(x)}, 1 \right]
\end{align*}\]

[compressible, pseudo-incompressible, anelastic]
NFT Integrator

Advection: MPDATA

Multidimensional Positive Definite Advection Transport Algorithm

$$\frac{\partial G \Phi}{\partial t} + \nabla \cdot (V \Phi) = GR$$

$$\Phi_i^{n+1} = A_i \left( \Phi_i^n + \frac{1}{2} \delta t R_i, V_i^{n+\frac{1}{2}}, G^n, G^{n+1} \right) + \frac{1}{2} \delta t R_i^{n+1}$$

Smolarkiewicz & Szmelter, JCP 2009, Acta Geo 2011

Helmholtz: from evolutionary form of the gas law

$$\frac{1}{\gamma} \nabla \cdot (\dot{G} \mathbf{v}^\nu) - \frac{1}{\xi} \frac{\phi_{\nu\nu}}{\rho^\nu-1} \left( \frac{1}{\rho^\nu} \nabla \cdot (\mathbf{q}^\nu \mathbf{v}^\nu) - \frac{1}{\rho^\nu} \nabla \cdot (\mathbf{q}^\nu \mathbf{v}^\nu) \right) - \frac{1}{\delta t \xi \phi_{\nu\nu}-1} (\mathbf{v}^\nu - \hat{\mathbf{v}}) = 0$$

Smolarkiewicz et. al. JCP 2016

$$\Rightarrow \frac{1}{\zeta} \nabla \cdot \zeta \left( \mathbf{v} - \mathbf{G}^T C \nabla \phi \right)$$

Implicit Integrator: GCR(k)

non-symmetric, preconditioned Krylov-subspace elliptic solver

For any initial guess, $\phi^0$, set $r^0 = \mathcal{L}(\phi^0) - \mathcal{R}, p^0 = \mathcal{P}^{-1} (r^0)$; then iterate:

For $n = 1, 2, \ldots$ until convergence

for $\nu = 0, \ldots, k - 1$

$$\beta = \frac{\langle r^\nu \mathcal{L}(p^\nu) \rangle}{\langle \mathcal{L}(p^\nu) \mathcal{L}(p^\nu) \rangle}$$

$$\phi_{\nu+1} = \phi^\nu + \beta p^\nu,$$

$$r_{\nu+1} = r^\nu + \beta \mathcal{L}(p^\nu),$$

exit if $\|r_{\nu+1}\| \leq \epsilon$,

$$\epsilon = \mathcal{P}^{-1} (r_{\nu+1}),$$

evaluate $\mathcal{L}(c) = \frac{1}{\rho^\nu} \nabla \cdot C \nabla c$,

for $l = 0, \ldots, \nu$

$$\alpha_l = \frac{\langle \mathcal{L}(c) \mathcal{L}(p^l) \rangle}{\langle \mathcal{L}(p^l) \mathcal{L}(p^l) \rangle},$$

$$p_{\nu+1} = p^\nu + \sum_l \alpha_l p^l,$$

$$\mathcal{L}(p_{\nu+1}) = \mathcal{L}(c) + \sum_l \alpha_l \mathcal{L}(c^l),$$

reset $[\phi, r, c, \mathcal{L}(c)]^k$ to $[\phi, r, c, \mathcal{L}(c)]^0$

Smolarkiewicz & Margolin, 2000
Edge based finite volume discretisation

Uses a co-located data arrangement

Median dual computational mesh
Finite volumes

Smolarkiewicz & Szmelter, Acta Geo 2011
Convective Planetary Boundary Layer

Smolarkiewicz et al JCP 2013

Vertical velocity in central xz cross section (top) and the horizontal plane (bottom): instantaneous solution for triangular prismatic mesh after ~13 eddy turnover times.

Schmidt & Schummann JFM 1989

Vertical profile of dimensionless resolved heat flux, temperature variance, and vertical velocity; for dimensionless height $z/z_i$.

Edge-based $T$  
Edge-based $C$  
EULAG  
SS LES  
Observation
Preconditioning

Elementary preconditioner, slower elliptic solver

Advanced Preconditioner, faster elliptic solver

\[ I \leftarrow \mathcal{P} \rightarrow \mathcal{L} \]

Choose \( \mathcal{P} \) close to \( \mathcal{L} \), but easier to solve.

Take only diagonal terms in \( C \): \( \mathcal{P} \approx \mathcal{L} \)

Invert \( \mathcal{P}_z \) implicitly, with \( \mathcal{P}_H \) lagged:

\[
\frac{\partial ee}{\partial \tilde{\tau}} = \mathcal{P}(e) - r \quad \Rightarrow \quad \frac{ee^{\mu+1} - ee^\mu}{\Delta \tilde{\tau}} = \mathcal{P}_H(e^{\mu}) + \mathcal{P}_z(e^{\mu+1}) - r
\]

\[
(I - \Delta \tilde{\tau} \mathcal{P}_z) e^{\mu+1} = e^\mu + \Delta \tilde{\tau} \left( \mathcal{P}_H(e^\mu) - r^{\nu+1} \right)
\]

For any initial guess, \( \phi^0 \), set \( r^0 = \mathcal{L}(\phi^0) - R \), \( p^0 = \mathcal{P}^{-1}(r^0) \); then iterate:

For \( n = 1, 2, \ldots \) until convergence

\[
\beta = \frac{\langle r^{\nu} \mathcal{L}(p^\nu) \rangle}{\langle \mathcal{L}(p^\nu) \mathcal{L}(p^\nu) \rangle},
\]

\[
\phi^{\nu+1} = \phi^\nu + \beta p^\nu,
\]

\[
r^{\nu+1} = r^\nu + \beta \mathcal{L}(p^\nu),
\]

exit if \( ||r^{\nu+1}|| \leq \epsilon \),

\[
e = \mathcal{P}^{-1}(r^{\nu+1}),
\]

evaluate \( \mathcal{L}(e) = \frac{1}{\rho^e} \nabla \cdot C \nabla e \),

for \( l = 0, \ldots, \nu \)

\[
\alpha_l = \frac{\langle \mathcal{L}(e) \mathcal{L}(p^l) \rangle}{\langle \mathcal{L}(p^l) \mathcal{L}(p^l) \rangle},
\]

\[
p^{\nu+1} = e + \sum_l \alpha_l p^l,
\]

\[
\mathcal{L}(p^{\nu+1}) = \mathcal{L}(e) + \sum_l \alpha_l \mathcal{L}(e^l),
\]

reset \( [\phi, r, e, \mathcal{L}(e)^0] \) to \( [\phi, r, e, \mathcal{L}(e)^0] \)

Smolarkiewicz & Margolin, 2000
Multigrid meshes using Atlas

Oct16

20

16

+4
Multigrid meshes using Atlas

Octahedral mesh has reasonably consistent structure

Latitude $n = 1 \ldots N$ has $20 + 4n$ longitudes
Multigrid meshes using Atlas

Remove every other latitude

Half the number of longitudes at the remaining latitudes

Reduces horizontal nodes by a factor of \( \sim 4 \)

Need to pick starting latitude

Coarse mesh is almost octahedral

All coarse mesh nodes coincide with nodes on the fine mesh
Multigrid meshes using Atlas

Octahedral 16 mesh: Remove odd latitudes

computational domain Single level of mesh coarsening
Multigrid meshes using Atlas

Octahedral 16 mesh: Remove odd latitudes

Computational domain Single level of mesh coarsening
**Multigrid Preconditioning**

V-Cycle: Solve on $h=1$, else cycle($e_h, r_h$)

- Smooth($e_h$)
- calculate residue: $rr_h = \mathcal{P}(e_h) - r_h$
- restrict $rr_h \rightarrow rr_{h-1}$: $rr_{h-1} = \mathcal{I}_{h-1}^h rr_h$
- initialise solution error: $ee_{h-1} [\equiv e - e']$
- cycle($ee_{h-1}, rr_{h-1}$)
- Interpolate solution error: $ee_h = \mathcal{I}_{h-1}^h ee_{h-1}$
- Add coarse grid error: $e_h = e_h - ee_h$

Smooth($e_h$)

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### V-Cycle

- Initialise
- Solve
- Interpolate
- Smooth
- Restriction

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Smother / Solver, varying number of iterations depending requirements

$$ (I - \Delta \tilde{\tau} \mathcal{P}_z) e^{\mu+1} = e^\mu + \Delta \tilde{\tau} \left( \mathcal{P}_H(e^\mu) - r^{\nu+1} \right) $$
Multigrid Preconditioning

Baroclinic Instability – 9 days.
Compressible Nonhydrostatic Equations.
Multigrid / implicit vertical preconditioner.

Multigrid:
Wall time ~ 4hrs
Elliptic solver iterations average – 3:0

Without multigrid:
Wall time ~ 23hrs
Elliptic solver iterations average – 60:7

Setup:
Octahedral 80 grid,
time-step 900s,
1/1 task/thread,
Conclusion

• NFT-MPDATA solvers based on an edge based finite volume method for unstructured grids provide a high quality results for a wide range of mesh types and is applicable for all scales atmospheric flows.

• Substantial efficiency gains have been achieved by introducing a horizontal multigrid preconditioner to a Krylov solver of Helmholtz equations forming a part of a global nonhydrostatic model.