Split-explicit methods and and local linear splitting

Oswald Knoth

Leibniz Institute for Tropospheric Research, Leipzig

3.10.2016

Contents

Time scales

Time scales in the atmosphere, flow

- Sound waves 340m/s
- Gravity waves 100m/s
- Wind speed 60m/s

Use time integration methods which use different time steps for different

$$y' = f(y) + g(y)$$

Compressible Euler equations

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \boldsymbol{\nabla}(\rho \mathbf{v}\mathbf{v}) - \boldsymbol{\nabla}\boldsymbol{\tau} = -\nabla p - \rho g \mathbf{k} - 2\boldsymbol{\Omega} \times (\rho \mathbf{v})$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0$$

$$\frac{\partial(\rho\theta)}{\partial t} + \nabla(\rho\theta\mathbf{v}) - \nabla(\rho \mathbf{K}_\theta \nabla\theta) = Q_\theta$$

$$\frac{\partial(\rho\mu)}{\partial t} + \nabla(\rho\mu\mathbf{v}) - \nabla(\rho \mathbf{K}_\mu \nabla\mu) = Q_\mu$$

$$p = \left(\frac{R_d \rho\theta}{p_0^\kappa}\right)^{1/(1-\kappa)}$$

- Pressure is determined by equation of state: $p = P(\rho, \theta, \mathbf{c})$
- Moist potential temperature θ
- In the dry case $\theta = T\left(\frac{p_0}{p}\right)^\kappa$ with $\kappa \equiv R_d/c_{pd} = \text{const.}$

Splitting nonlinear equation

$$\frac{\partial\rho}{\partial t} = -\frac{\partial\rho u}{\partial x} - \frac{\partial\rho w}{\partial z}$$
$$\frac{\partial\rho u}{\partial t} = -\frac{\partial\rho uu}{\partial x} - \frac{\partial\rho wu}{\partial z} - \frac{\partial p}{\partial x}$$
$$\frac{\partial\rho w}{\partial t} = -\frac{\partial\rho uw}{\partial x} - \frac{\partial\rho ww}{\partial z} - \frac{\partial p}{\partial z} - \rho g$$
$$\frac{\partial\rho\theta}{\partial t} = -\frac{\partial\rho u\theta}{\partial x} - \frac{\partial\rho w\theta}{\partial z}$$

- $$\dot{y} = F(y, y)$$

Splitting nonlinear "linearized" equation

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho w}{\partial z} \\
\frac{\partial \rho u}{\partial t} &= -\frac{\partial \rho u u}{\partial x} - \frac{\partial \rho w u}{\partial z} - \frac{\partial p}{\partial \rho\theta}\frac{\partial \rho\theta}{\partial x} \\
\frac{\partial \rho w}{\partial t} &= -\frac{\partial \rho u w}{\partial x} - \frac{\partial \rho w w}{\partial z} - \frac{\partial p}{\partial \rho\theta}\frac{\partial \rho\theta}{\partial z} - \rho g \\
\frac{\partial \rho\theta}{\partial t} &= -\frac{\partial \rho u\theta}{\partial x} - \frac{\partial \rho w\theta}{\partial z}
\end{aligned}
$$

- $$\dot{y} = F(y) + A(y)y$$

Splitting linearized equation

- The approximate, quasi-Boussinesq linearized equations

$$
\begin{aligned}
u_t &= -Uu_x - c_s p_x \\
w_t &= -Uw_x - c_s p_z - N\theta \\
\theta_t &= -U\theta_x - Nw \\
p_t &= -Up_x - c_s(u_x + w_z)
\end{aligned}
$$

- One dimensional acoustic advection system

$$
\begin{aligned}
u_t &= -Uu_x - c_s p_x \\
p_t &= -Up_x - c_s u_x
\end{aligned}
$$

Split system

$$\dot{u} = f_u(u) + g_p(p)$$
$$\dot{p} = f_p(p) + g_u(u)$$

- Slow system

$$\dot{u} = f_u(u)$$
$$\dot{p} = f_p(p)$$

- Integrated wit a Runge-Kutta method
- Fast system

$$\dot{u} = g_p(p)$$
$$\dot{p} = g_u(u)$$

- Integrated with symplectic Euler (Forward-Backward)

Underlying Runge-Kutta

- Wicker and Skamarock (MWR 2002) used a three-stage Runge-Kutta method as slow integrator:

$$
\begin{aligned}
u^{n+1/3} &= u^n + \frac{\Delta t}{3} f_u(u^n) \\
p^{n+1/3} &= p^n + \frac{\Delta t}{3} f_p(p^n) \\
u^{n+1/2} &= u^n + \frac{\Delta t}{2} f_u(u^{n+1/3}) \\
p^{n+1/2} &= p^n + \frac{\Delta t}{2} f_p(p^{n+1/3}) \\
u^{n+1} &= u^n + \Delta t f_u(u^{n+1/2}) \\
p^{n+1} &= p^n + \Delta t f_p(p^{n+1/2})
\end{aligned}
$$

Split-explicit

- Resulting splitting scheme:

$$
\begin{aligned}
& u = u^n, \qquad p = p^n \\
& \text{for } k = 1 : n_s/3 \\
& \qquad u = u + \Delta\tau g_u(p) + \Delta\tau f_u(u^n) \\
& \qquad p = p + \Delta\tau g_p(u) + \Delta\tau f_p(p^n) \\
& \text{end} \\
& u^{n+1/3} = u, \qquad p^{n+1/3} = p, \qquad u = u^n, \qquad p = p^n \\
& \text{for } k = 1 : n_s/2 \\
& \qquad u = u + \Delta\tau g_u(p) + \Delta\tau f_u(u^{n+1/3}) \\
& \qquad p = p + \Delta\tau g_p(u) + \Delta\tau f_p(p^{n+1/3}) \\
& \text{end} \\
& u^{n+1/2} = u, \qquad p^{n+1/2} = p, \qquad u = u^n, \qquad p = p^n \\
& \text{for } k = 1 : n_s \\
& \qquad u = u + \Delta\tau g_u(p) + \Delta\tau f_u(u^{n+1/2}) \\
& \qquad p = p + \Delta\tau g_p(u) + \Delta\tau f_p(p^{n+1/2}) \\
& \text{end}
\end{aligned}
$$

General split-explicit methods

- Assume that we can solve the fast part of

$$\dot{y} = f(y) + g(y)$$

"exact"

- Then a split Runge–Kutta method reads:

$$\begin{aligned} Z_{ni}(0) &= y_n \\ \dot{Z}_{ni}(\tau) &= \frac{1}{c_i} \sum_{j=1}^{i-1} a_{ij} f(Y_{nj}) + g(Z_{ni}(\tau)) \\ Y_{ni} &= Z_{ni}(c_i h) \\ y_{n+1} &= Y_{n,s+1} \end{aligned}$$

- For the nonlinear case

$$\dot{Z}_{ni}(\tau) = \frac{1}{c_i} \sum_{j=1}^{i-1} a_{ij} F(Y_{nj}, Z_{ni}(\tau))$$

The method

- We generalize the exact integration procedure in two directions:
  - arbitrary starting points based on preceding stages

$$Z_{ni}(t_n + \epsilon_{i-1}\Delta t) = y_n + \sum_{j=1}^{i-1} \alpha_{ij}(Y_{nj} - y_n)$$

  - increments in the constant term F based on preceeding stages

$$\dot{Z}_{ni}(\tau) = \frac{1}{\delta_i}\left(\sum_{j=1}^{i-1} \gamma_{ij}(Y_{nj} - y_n) + \sum_{j=1}^{i-1} \beta_{ij} f(Y_{nj})\right) + g(Z_{ni}(\tau))$$

- Intermediate time intervals for local integration

$$\tau \in [y_n + \epsilon_{i-1}\Delta t, y_n + \epsilon_i \Delta t]$$

- New set of parameters, includes standard Runge-Kutta methods

Phi function and linear equations

Other names: Exponential time differencing (ETD)

$$\dot{y} = N(y) + Ly, \tag{1}$$

Assume

$$N(y(t+h)) = \sum_{k=0}^{m-1} \frac{h^k}{k!} N_k$$

then (1) has the solution

$$y(t+h) = \sum_{k=1}^{m} h^k \phi_k(hL) N_{k-1} + \exp(hL) y(t)$$

with

$$\phi_k(z) = \frac{\phi_{k-1}(z) - 1/(k-1)!}{z}$$

The split explicit method

Apply the split-explicit method to (1)

$$
\begin{aligned}
Y_1 &= y_n \\
Y_2 &= exp(1/3hL)y_n + h\phi_1(1/3hL)N(Y_1) \\
Y_3 &= exp(1/2hL)y_n + h\phi_1(1/2hL)N(Y_2) \\
y_{n+1} &= exp(hL)y_n + h\phi_1(hL)N(Y_3)
\end{aligned}
$$

Exponential Rosenbrock-like methdos

Now use with

$$J_n = f'(y_n) + g'(y_n)$$

$$N(y) = f(y) + g(y) - J_n y$$
$$L = J_n$$

- New coefficients
- Partioned methods or W-methods with $J_n = g'(y_n)$

Idea

- Splitting by adding and subtracting a linear term
- $y' = F(y) = Ly + (F(Y) - Ly) = Ly + f(y)$
- Split the linear term again in two parts $L = L_E + L_I$
- A sub step in the split explicit method reads
  $Z' = L_E Z + L_I Z + r$
- Apply an implicit-explicit method (IMEX) to this linear equation

IMEX-Methods

- Partitioned Runge-Kutta methods for additive splitting
- One method explicit, second diagonal implicit (DIRK)

$$\begin{array}{c|ccc} & & & \\ c_2 & a_{21} & & \\ c_3 & a_{31} & a_{32} & \\ \hline & b_1 & b_2 & b_3 \end{array} \qquad \begin{array}{c|ccc} \bar{c}_1 & \bar{\gamma} & & \\ \bar{c}_2 & \bar{a}_{21} & \bar{\gamma} & \\ \bar{c}_3 & \bar{a}_{31} & \bar{a}_{32} & \bar{\gamma} \\ \hline & \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{array}$$

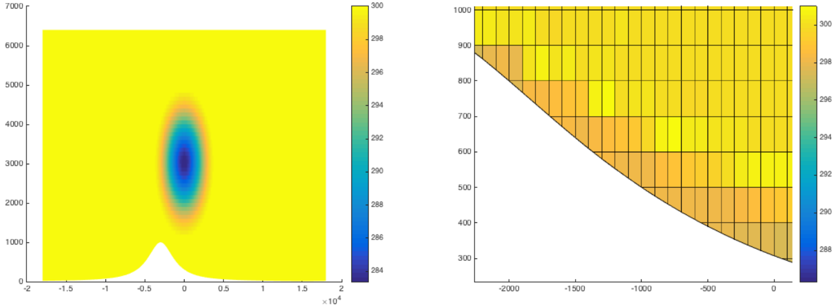
$$Z_i = Z_0 + \Delta\tau \sum_{j=1}^{i-1} a_{ij} L_E Z_j + \Delta\tau \sum_{j=1}^{i-1} \bar{a}_{ij} L_I Z_j + \Delta\tau\hat{\gamma} L_I Z_i$$

$$(I - \Delta\tau\hat{\gamma} L_I) Z_i = Z_0 + \Delta\tau L_E \sum_{j=1}^{i-1} a_{ij} Z_j + \Delta\tau L_I \sum_{j=1}^{i-1} \bar{a}_{ij} Z_j$$

- Per step two matrix vector multiplications and one linear solve

Straka-test with orography

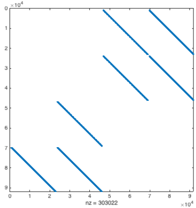
- Cut cell grid
- 36000 m in the horizontal, 6400 m in the vertical, grid size 100 m

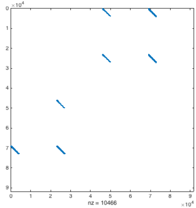


- Linear part: Acoustic for the whole domain and advection near cut cells
- Linear implicit part: Acoustic and advection near cut cells

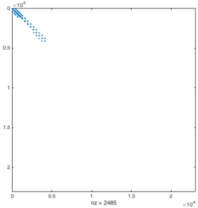
Fill in structure

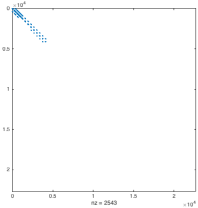
- Fill in explicit versus implicit part for linear acoustic



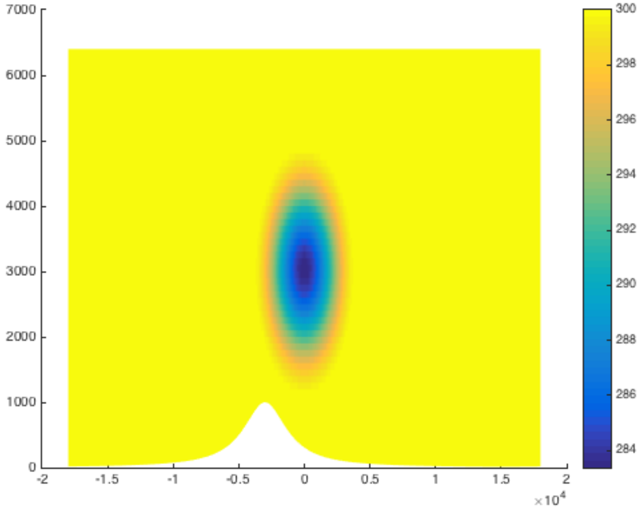


- Fill in implicit part for advection, scalar and vertical wind

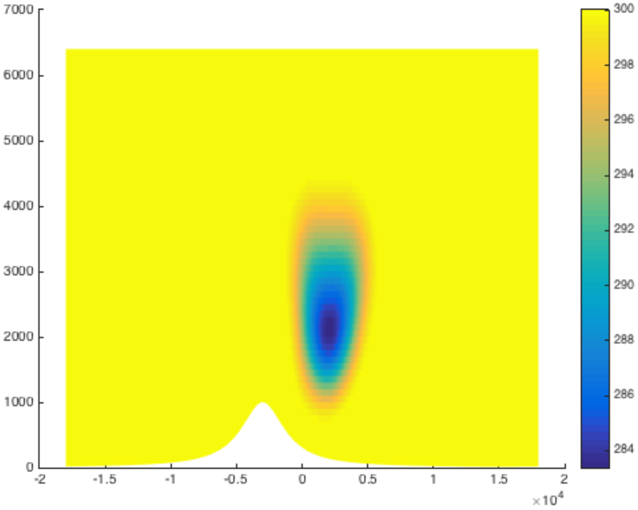




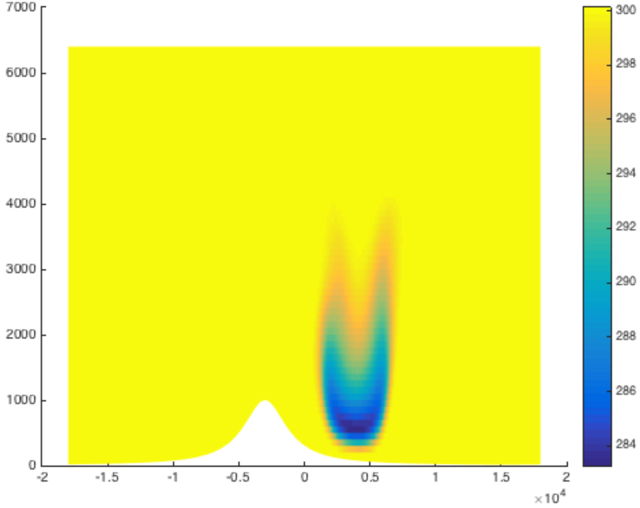
Cold bubble


7000
6000
5000
4000
3000
2000
1000
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
×10^4
300
298
296
294
292
290
288
286
284

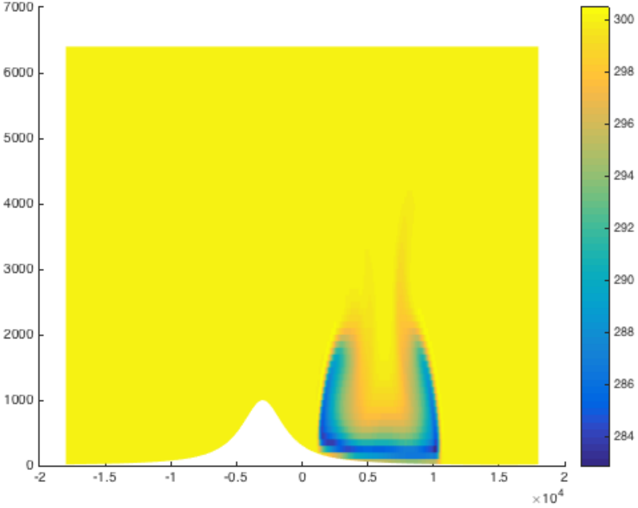
Cold bubble


7000
6000
5000
4000
3000
2000
1000
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
$\times 10^4$
300
298
296
294
292
290
288
286
284

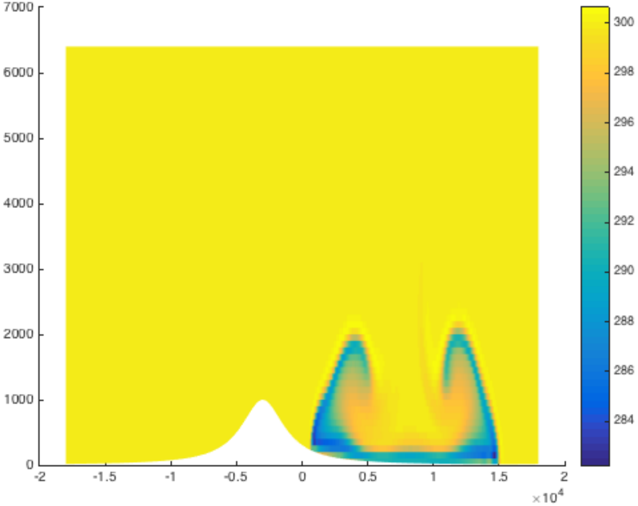
Cold bubble


7000
6000
5000
4000
3000
2000
1000
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
×10^4
300
298
296
294
292
290
288
286
284

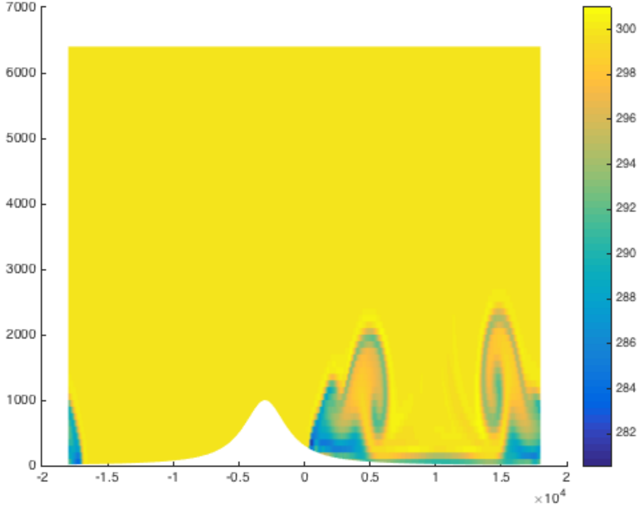
Cold bubble


7000
6000
5000
4000
3000
2000
1000
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
×10^4
300
298
296
294
292
290
288
286
284

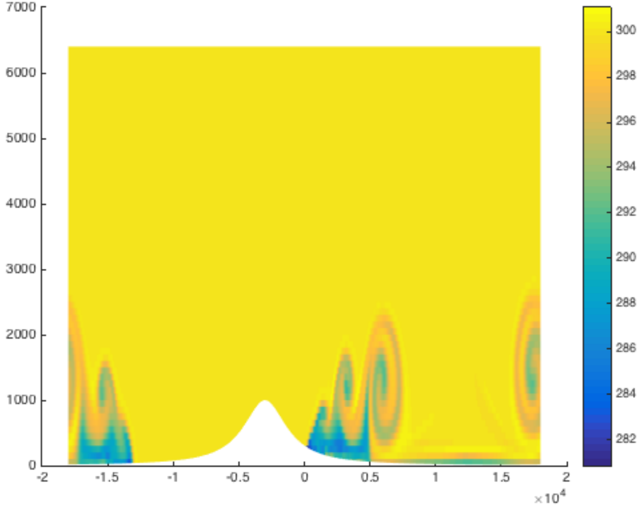
Cold bubble


7000
6000
5000
4000
3000
2000
1000
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
$\times 10^4$
300
298
296
294
292
290
288
286
284

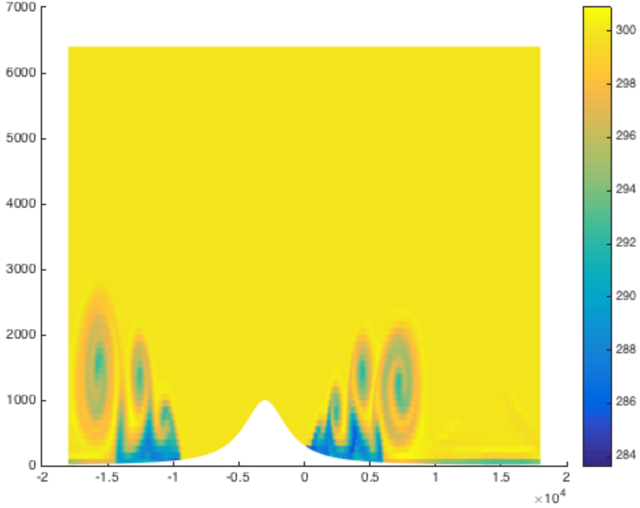
Cold bubble


7000
6000
5000
4000
3000
2000
1000
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
×10^4
300
298
296
294
292
290
288
286
284
282

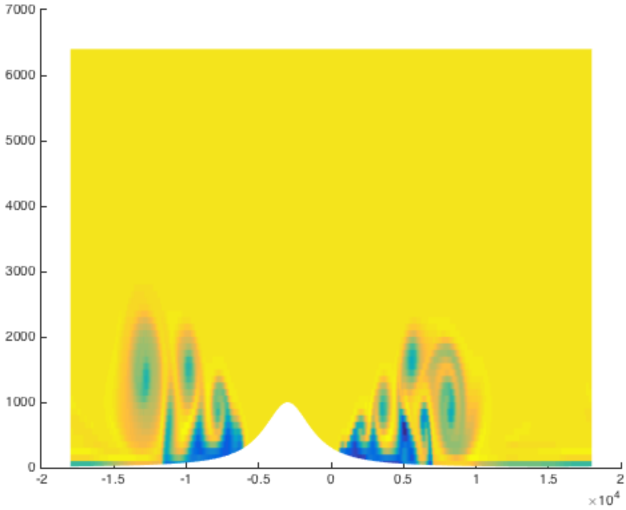
Cold bubble


7000
6000
5000
4000
3000
2000
1000
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
$\times 10^4$
300
298
296
294
292
290
288
286
284
282

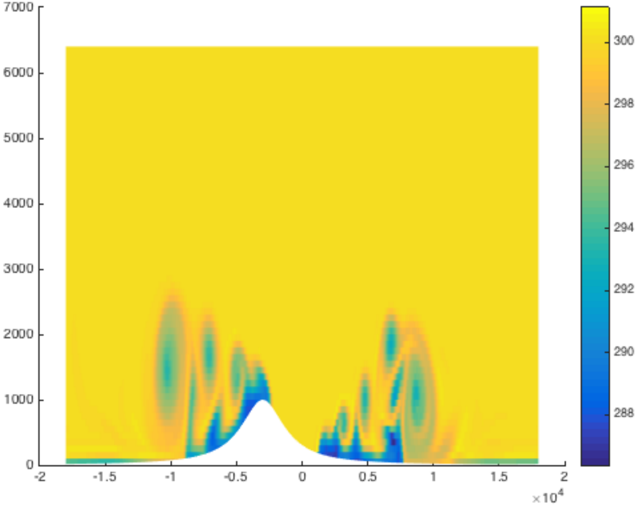
Cold bubble


7000
6000
5000
4000
3000
2000
1000
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
×10^4
300
298
296
294
292
290
288
286
284

Cold bubble



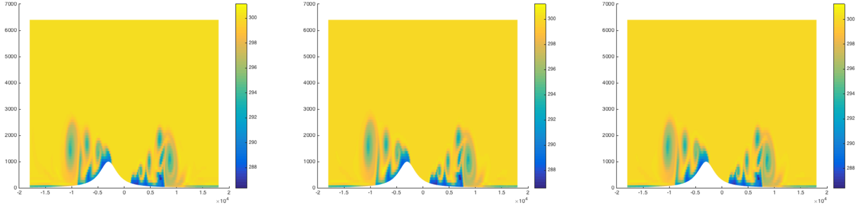
Cold bubble



7000
6000
5000
4000
3000
2000
1000
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
×10^4
300
298
296
294
292
290
288

Cold bubble

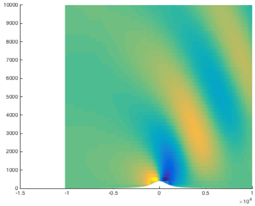
- Cold bubble with large time step of 1 s, 2 s, and 3 s.

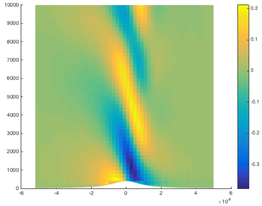


dt	Small time steps			
1	3	3	2	4
2	6	6	3	8
3	8	8	4	11

Example of Gallus/Klemp

- Flow over hill of 400m of a stratified flow
- Nonhydrostatic case $\Delta x = 200$m and $\Delta x = 100$m
- Hydrostatic case $\Delta x = 2000$m and $\Delta x = 100$
- First case only cut cells implicit, $\Delta t = 20$s
- Number of small time steps, 23, 21, 11,32
- Second case cut cells and vertical direction for acoustic, $\Delta t = 100$s
- Number of small time steps 9, 8, 4, 12





Conclusions

- Split-explicit methods are a special exponential integrator
- Room for new methods
- Can be used for cut cell grids with small additional overhead
- Numerical examples demonstrate theory

Future work

- Automatic choice of implicit part
- Efficient implementation for parallel application
  - Organize computation for sub vectors
  - New data structure for exchange for data
- Tests for global examples