Dimension Splitting Errors and a Long Time-Step Multi-Dimensional Scheme for Atmospheric Transport

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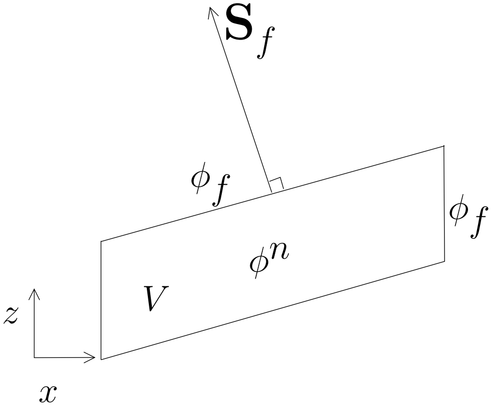
Motivation

- With cubed-sphere grid, Met-Office no longer need semi-Lagrangian for large Courant number over the poles
- Mass conservation
- Still need Courant number > 1 in vertical
- Options for transport
  - Method of lines versus flux-form semi-Lagrangian (Forward in time)
  - Implicit, Explicit or HEVI
  - Dimensionally split or multi-dimensional

Finite Volume Advection

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot (\mathbf{u}\phi) - \frac{1}{V}\sum_{f \in \text{faces}} \phi_f \mathbf{u}_f \cdot \mathbf{S}_f$$

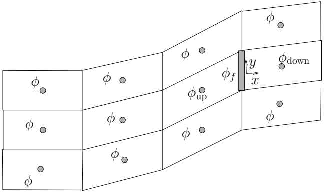


How to evaluate ϕ_f between times n and $n+1$

Method of Lines - Spatial Discretisation

- Method of lines - separate discretisation for space and time
- At every instant, interpolate from surrounding cell values of ϕ onto the face using an upwind-biased stencil:

27 cells in 3D, 12 in 2D:



Fit the polynomial:

$$a + bx + cx^2 + dx^3 + ey + fxy + gx^2y$$

Second order assumptions:
- Cell averages = cell centre values
- Face averages = face centre values

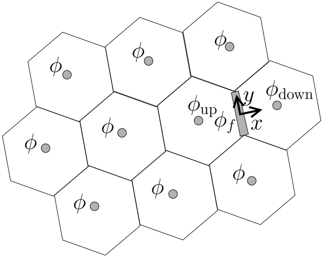
Use a least squares fit to find weights w_c for every cell in the stencil:

$$\phi_f = \phi_{up} + \sum_{c \in \text{stencil}} w_c \phi_c$$

Weights remain fixed every time-step

Method of Lines - Spatial Discretisation

10 cells in 2D:



Fit the polynomial:

$$a + bx + cx^2 + dx^3 + \\ ey + fxy + gx^2y$$

Method of Lines - can use Runge-Kutta Explicit Time Discretisation

Heun (2-stage, 2nd order RK):

$$\begin{aligned}
\phi' &= \phi^n - \frac{\Delta t}{V} \sum_{f \in c} \left(\phi^n_{\text{up}} + \sum_{c \in \text{stencil}} w_c \phi^n_c \right) \mathbf{u}_f \cdot \mathbf{S}_f \\
\phi^{n+1} &= \frac{1}{2}\phi^n + \frac{1}{2}\phi' - \frac{\Delta t}{2V} \sum_{f \in c} \left(\phi'_{\text{up}} + \sum_{c \in \text{stencil}} w_c \phi'_c \right) \mathbf{u}_f \cdot \mathbf{S}_f
\end{aligned}$$

Method of Lines - Implicit Time Discretisation

Crank-Nicholson (as explicit correction of 1st-order upwind):

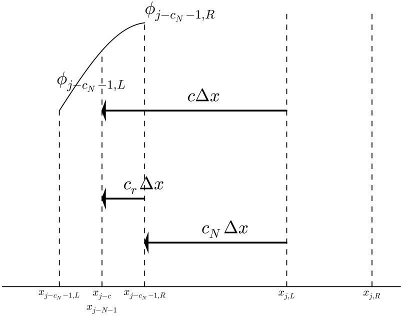
$$\begin{aligned}
\phi' &= \phi^n - \frac{\Delta t}{2V} \sum_{f \in c} \left(\phi^n_{\text{up}} + \phi'_{\text{up}} + 2 \sum_{c \in \text{stencil}} w_c \phi^n_c \right) \mathbf{u}_f \cdot \mathbf{S}_f \\
\phi^{n+1} &= \phi^n - \frac{\Delta t}{2V} \sum_{f \in c} \left(\phi^n_{\text{up}} + \sum_{c \in \text{stencil}} w_c \phi^n_c + \phi^{n+1}_{\text{up}} + \sum_{c \in \text{stencil}} w_c \phi'_c \right) \mathbf{u}_f \cdot \mathbf{S}_f
\end{aligned}$$

Method of Lines - HEVI

If the faces of cell c are separated into horizontal and vertical faces, this can be made HEVI:

$$
\begin{aligned}
\phi' &= \phi^n - \frac{\Delta t}{2V} \sum_{f \text{ vertical}} \left(\phi^n_{\text{up}} + \phi'_{\text{up}} + 2 \sum_{c \in \text{stencil}} w_c \phi^n_c \right) \mathbf{u}_f \cdot \mathbf{S}_f \\
&- \frac{\Delta t}{V} \sum_{f \text{ horizontal}} \left(\phi^n_{\text{up}} + \sum_{c \in \text{stencil}} w_c \phi^n_c \right) \mathbf{u}_f \cdot \mathbf{S}_f \\
\phi^{n+1} &= \phi^n - \frac{\Delta t}{2V} \sum_{f \text{ vertical}} \left(\phi^n_{\text{up}} + \sum_{c \in \text{stencil}} w_c \phi^n_c + \phi^{n+1}_{\text{up}} + \sum_{c \in \text{stencil}} w_c \phi'_c \right) \mathbf{u}_f \cdot \mathbf{S}_f \\
&- \frac{\Delta t}{2V} \sum_{f \text{ horizontal}} \left(\phi^n_{\text{up}} + \sum_{c \in \text{stencil}} w_c \phi^n_c + \phi'_{\text{up}} + \sum_{c \in \text{stencil}} w_c \phi'_c \right) \mathbf{u}_f \cdot \mathbf{S}_f
\end{aligned}
$$

Dimensionally split Flux form semi-Lagrangian



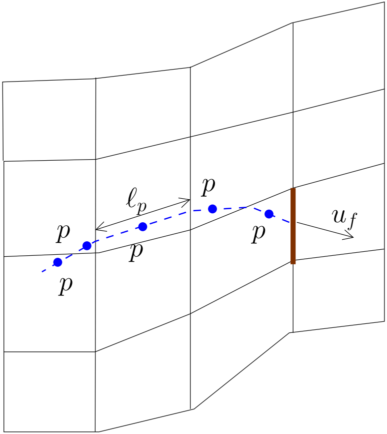
- PPM (Piecewise parabolic method Colella and Woodward, 1984) in each direction separately
- No monotonicity constraints
- Long time-steps
  - sum contributions from whole cells between face and departure point

- COSMIC splitting (Leonard et al., 1996)

$$\phi_{ij}^{(n+1)} = \phi_{ij}^{n} + X_C\left(\phi_{ij}^{(n)} + \frac{1}{2}Y_A\left(\phi_{ij}^{(n)}\right)\right) + Y_C\left(\phi_{ij}^{(n)} + \frac{1}{2}X_A\left(\phi_{ij}^{(n)}\right)\right)$$

Multi-dimensional Flux form semi-Lagrangian

Integrate along a single trajectory for each face to find ϕ_f, the average tracer swept through face f



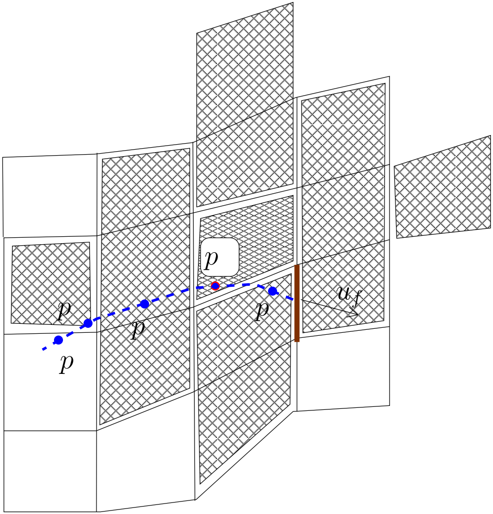
- $$\phi_f = \sum_{p \in t} \phi_p \ell_p$$

  Where ℓ_p is the length of the part of the trajectory associated with point p
- Interpolate ϕ onto all of the points p using a cubic polynomial with terms:

  $$a + bx + cx^2 + dx^3 + ey + fxy + gx^2y + hy^2 + ixy^2 + jy^3$$
- The points p are mid-way along the part of the trajectory in each cell

Multi-dimensional Flux form semi-Lagrangian

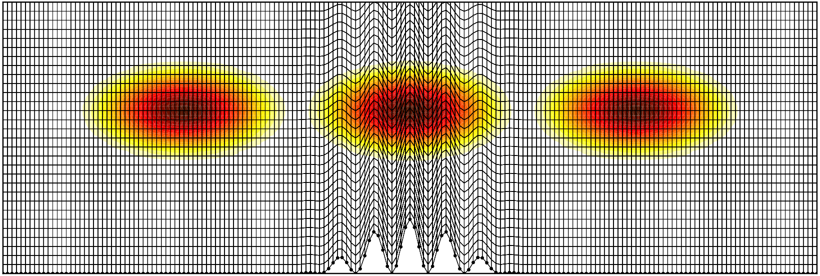
Stencil for interpolating onto each trajectory point, p;



- Trajectory points, stencils and interpolation weights re-calculated every time the wind changes
- Very expensive

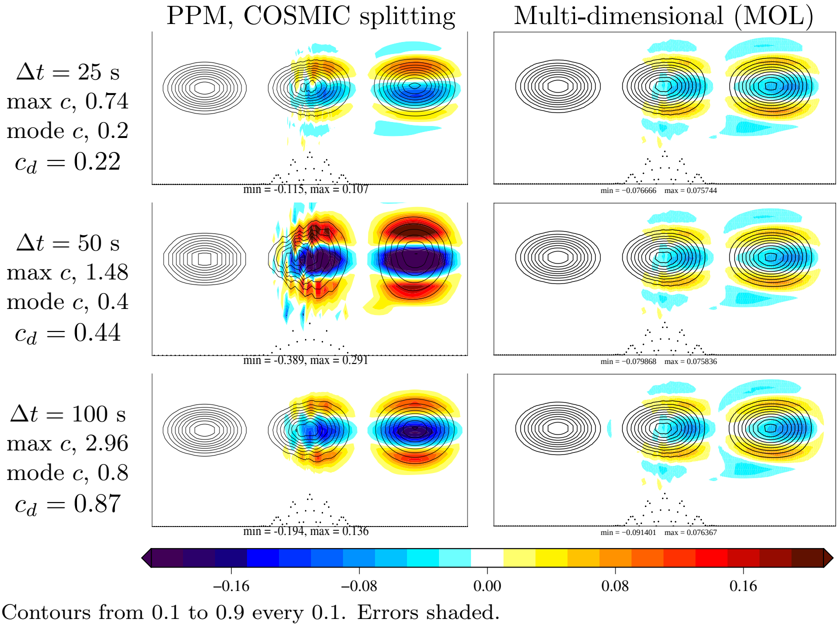
Test Case: Horizontal Advection Over Orography

Schär et al, MWR, 2002:

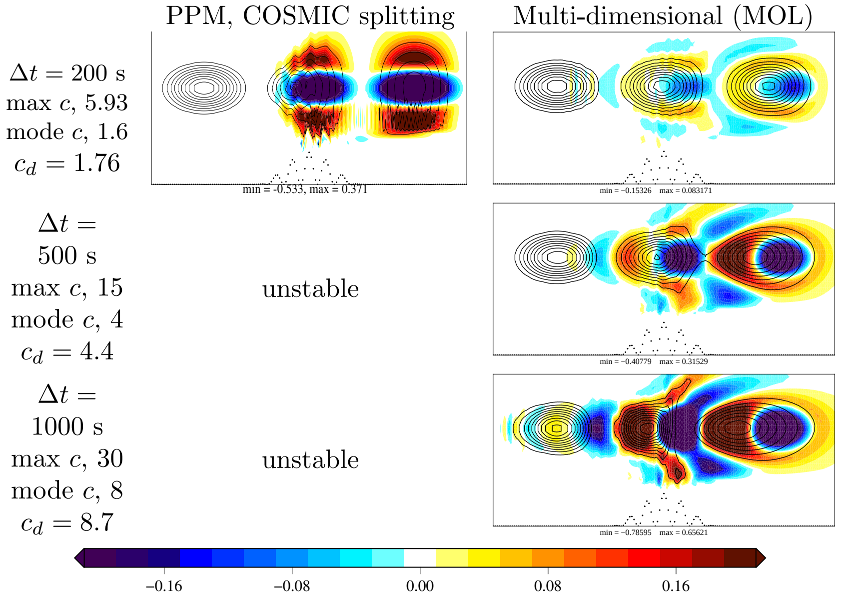


$t = 0$ $t = 5000s$ $t = 10000s$

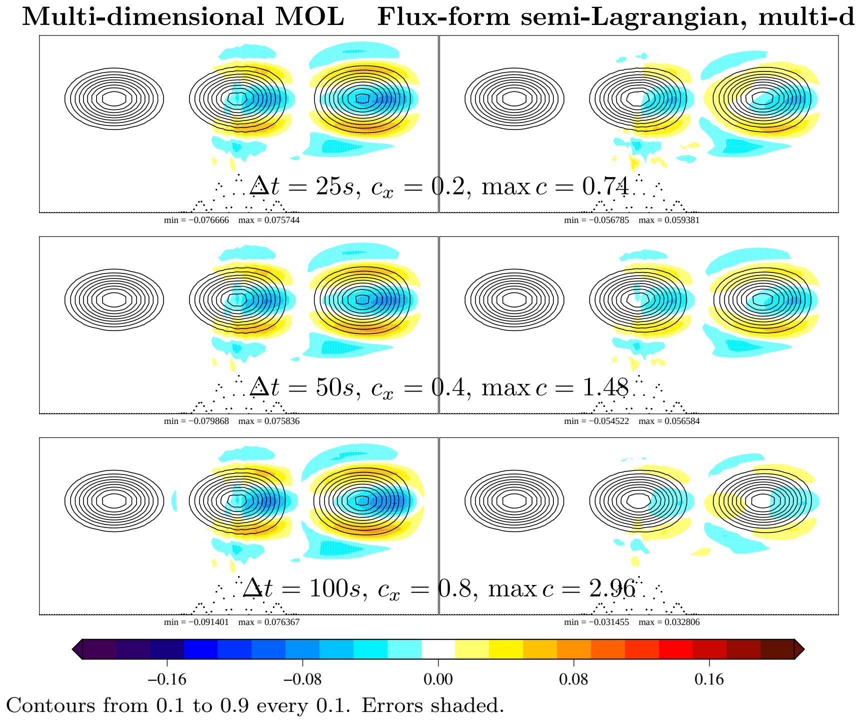
$\Delta x = 1km$, $\Delta z = 500m$, $u = 10m/s$, mountain height, $h_m = 3km$, tracer, $50km \times 6km$



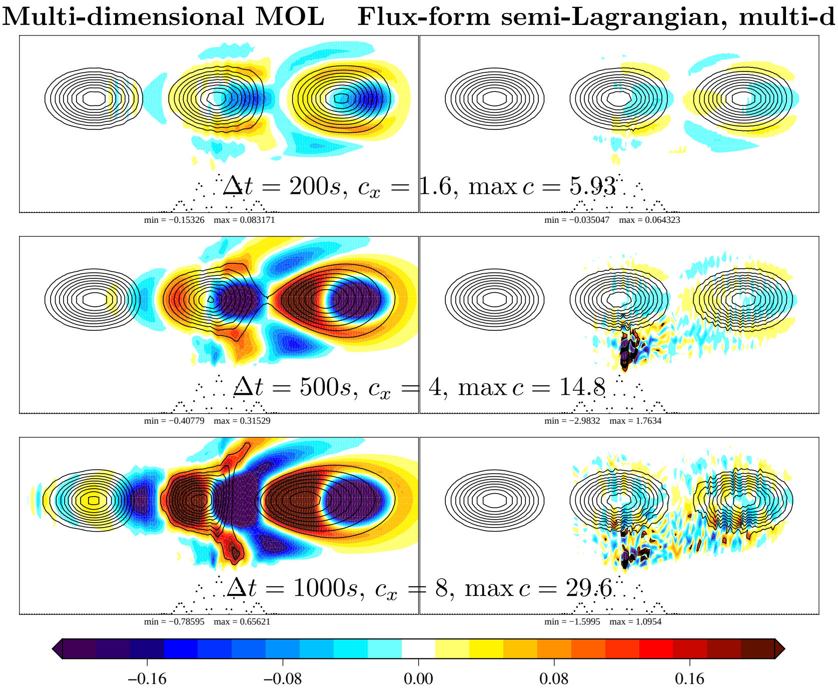
PPM, COSMIC splitting
Multi-dimensional (MOL)
$\Delta t = 25$ s
max c, 0.74
mode c, 0.2
$c_d = 0.22$
min = -0.115, max = 0.107
min = −0.076666 max = 0.075744
$\Delta t = 50$ s
max c, 1.48
mode c, 0.4
$c_d = 0.44$
min = -0.389, max = 0.291
min = −0.079868 max = 0.075836
$\Delta t = 100$ s
max c, 2.96
mode c, 0.8
$c_d = 0.87$
min = -0.194, max = 0.136
min = −0.091401 max = 0.076367
−0.16
−0.08
0.00
0.08
0.16
Contours from 0.1 to 0.9 every 0.1. Errors shaded.



Contours from 0.1 to 0.9 every 0.1. Errors shaded.



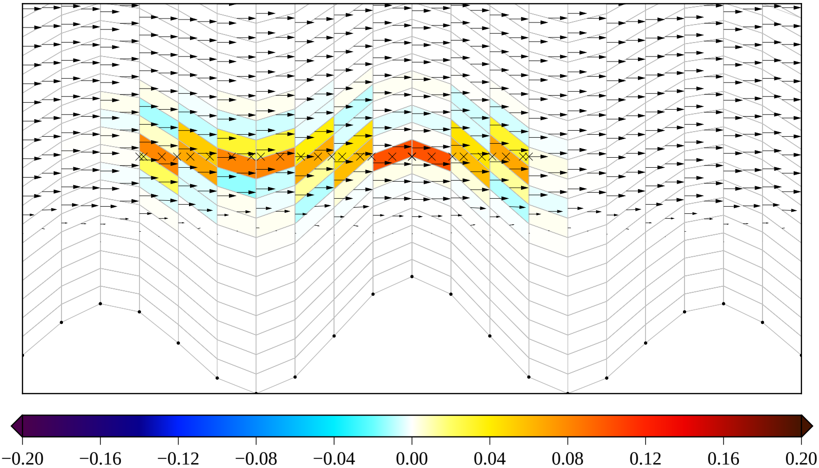
Multi-dimensional MOL
Flux-form semi-Lagrangian, multi-d
$\Delta t = 25s,\ c_x = 0.2,\ \max c = 0.74$
min = −0.076666 max = 0.075744
min = −0.056785 max = 0.059381
$\Delta t = 50s,\ c_x = 0.4,\ \max c = 1.48$
min = −0.079868 max = 0.075836
min = −0.054522 max = 0.056584
$\Delta t = 100s,\ c_x = 0.8,\ \max c = 2.96$
min = −0.091401 max = 0.076367
min = −0.031455 max = 0.032806
−0.16
−0.08
0.00
0.08
0.16
Contours from 0.1 to 0.9 every 0.1. Errors shaded.



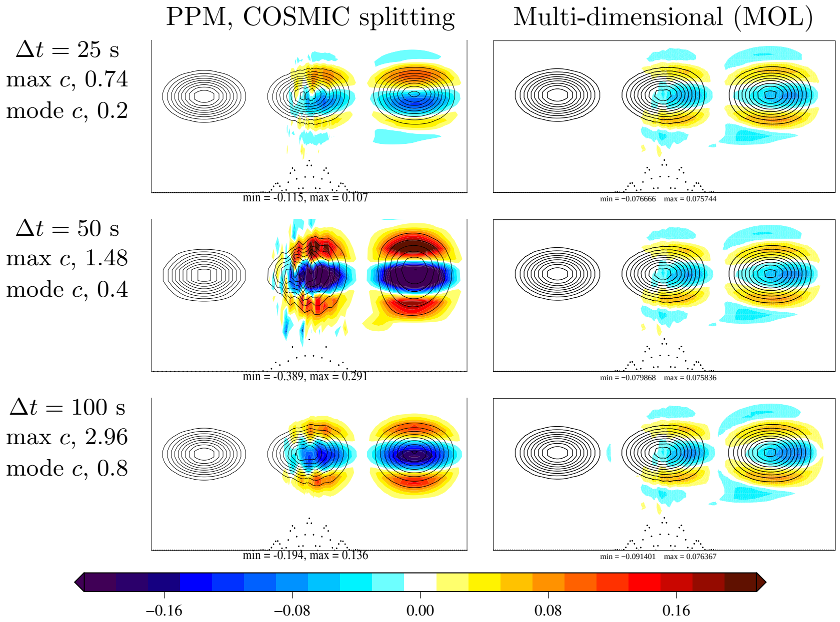
Contours from 0.1 to 0.9 every 0.1. Errors shaded.

Flux Form semi-Lagrangian Stencil over Orography

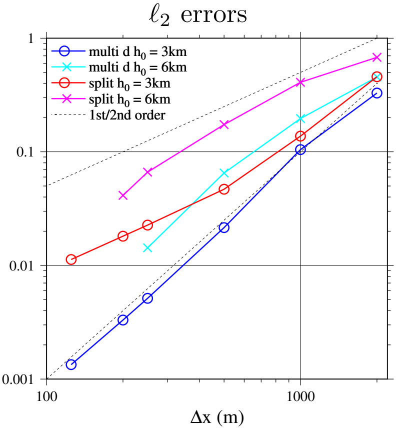
Points and weights for integrating along a trajectory. $c_x = 8$, $\max c = 29.6$

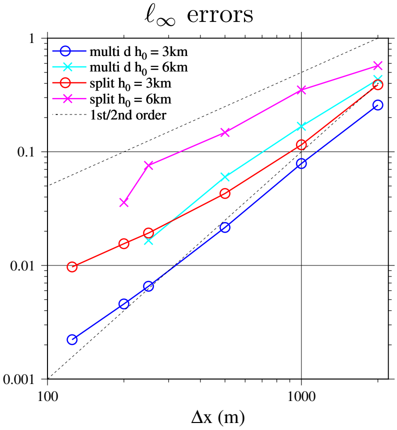


Raise the mountain to $h_0 = 6$km



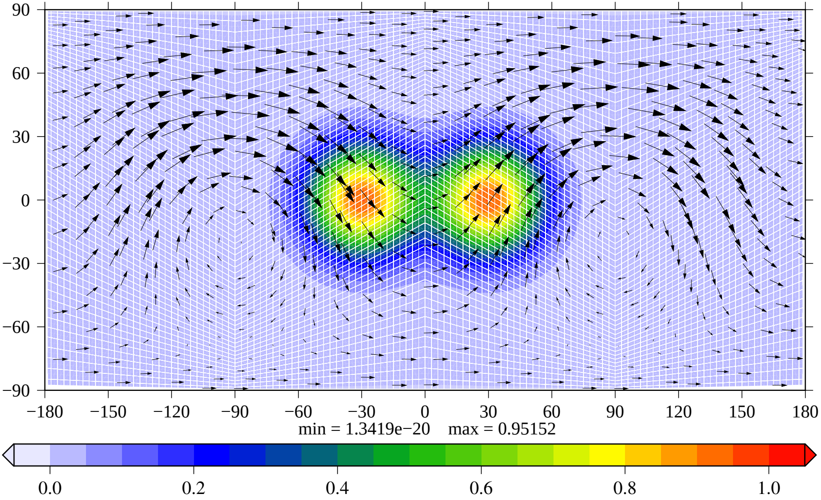
Convergence with Resolution Over a Mountain





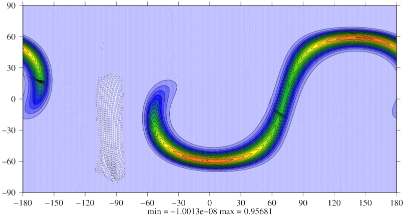
Deformational Flow on a Plane

Using a non-orthogonal grid:

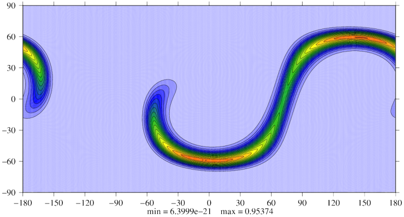


Results using 480×240 cells, $t = 1$ time unit

PPM with COSMIC splitting

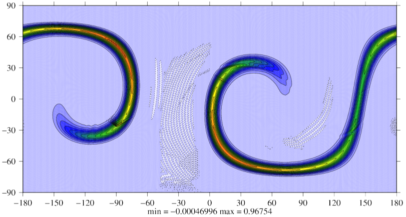


Multi-dimensional

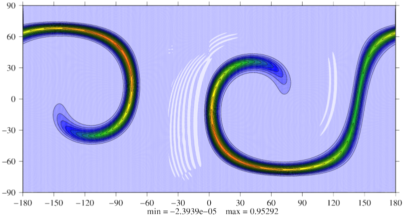


Results using 480×240 cells, $t = 2$ time unit

PPM with COSMIC splitting

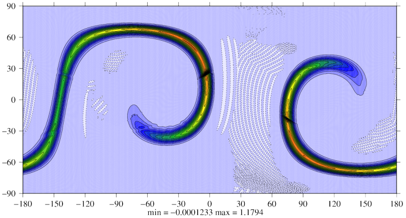


Multi-dimensional

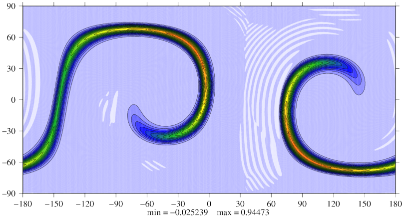


Results using 480×240 cells, $t = 3$ time unit

PPM with COSMIC splitting

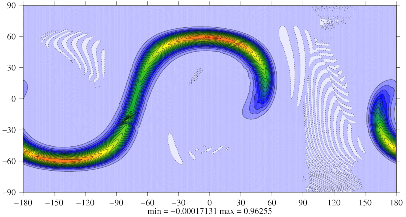


Multi-dimensional

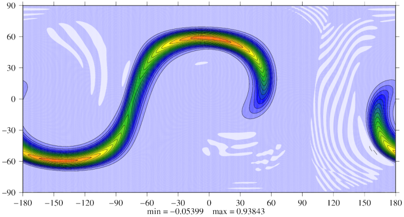


Results using 480×240 cells, $t = 4$ time unit

PPM with COSMIC splitting

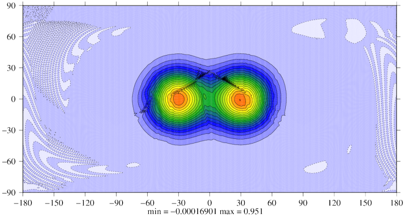


Multi-dimensional

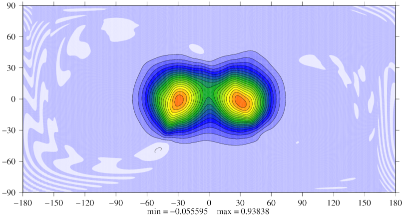


Results using 480×240 cells, $t = 5$ time unit

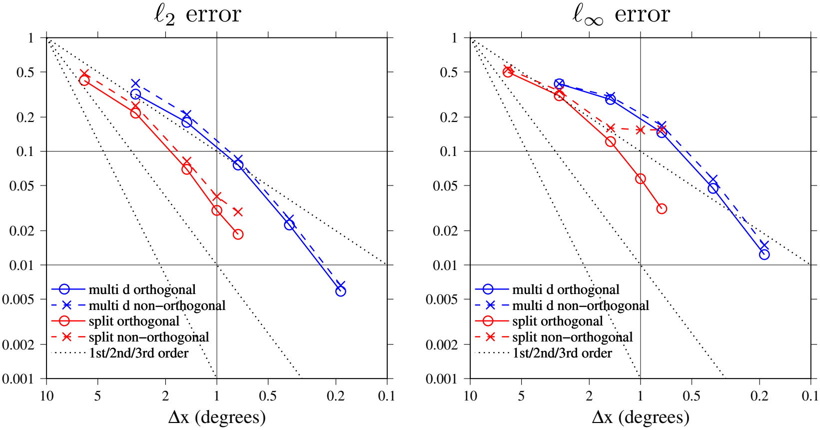
PPM with COSMIC splitting



Multi-dimensional



Convergence with Resolution



Conclusions

Multi-dimensional flux form semi-Lagrangian

- Very expensive
  - Cost proportional to Courant number
  - $\text{int}(c)+1$, 10×13 (in 2D) matrix inversions per face per time step for 2nd order accuracy with cubic interpolation
  - Computational geometry to find points in each cell along trajectory
- Stable for very long time-steps (c up to 30)
- Accurate for long time-step ($c \leq 3$)

Conclusions

Method of lines with Crank-Nicholson time-stepping

- Cost not strongly dependent on time-step
- Accurate on orthogonal and non-orthogonal grids
- Stable for very long time-steps (c up to 30) using deferred correction on 1st order upwind
- Phase lagging for $c > 5$ but phase speed $\nrightarrow 0$ even for c up to 30

PPM with COSMIC splitting

- More efficient and accurate on orthogonal grids
- Inaccuracies on non-orthogonal grids where non-orthogonality changes direction
  - excellent away from cubed-sphere edges
  - problems at cubed-sphere edges and over steep terrain (using terrain following grid)

P. Colella and P. Woodward. The piecewise parabolic method (PPM) for gas-dynamical simulations. *J. Comput. Phys.*, 1984.

B. Leonard, A. Lock, and M. MacVean. Conservative explicit unrestricted-time-step multidimensional constancy-preserving advection schemes. *Mon. Wea. Rev.*, 124(11):2585–2606, 1996.