A finite-volume module for cloud-resolving simulations of global atmospheric flows

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FVM extended to moist dynamics (with focus on coupling) ➔ synthesis of “dry” FVM [Piotr Smolarkiewicz, Willem Deconinck, Mats Hamrud, George Mozdzynski, Christian Kühnlein, Joanna Szmelter, Nils Wedi (JCP, 2016)] and “moist” schemes advanced in [Wojciech Grabowski & Piotr Smolarkiewicz (MWR, 1990, 1996, 2002)]

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Governing (compressible Euler) equations, a physically intuitive form:

\[
\begin{align*}
\frac{d \varrho}{dt} & = -\varrho \nabla \cdot \mathbf{u} = 0 , \\
\frac{d \mathbf{u}}{dt} & = -\frac{\theta(1 + q_v/\epsilon)}{\theta_0 (1 + q_t)} \nabla \varphi - \frac{g}{\theta_a} \left( \theta' + \theta_a (\varepsilon q_v' - q_c - q_p) \right) + \mathbf{D} , \\
\frac{d \theta'}{dt} & = -\mathbf{u} \cdot \nabla \theta_a + \frac{L \theta}{c_p T} (C_d + E_p) + \mathcal{H} , \\
\frac{d q_v}{dt} & = -C_d - E_p + D^{q_v} , \\
\frac{d q_c}{dt} & = C_d - A_p - C_p + D^{q_c} , \\
\frac{d q_p}{dt} & = A_p + C_p + E_p + D^{q_p} - \frac{1}{\varrho} \nabla \cdot (\varrho \mathbf{u} \cdot q_p) .
\end{align*}
\]

\[
\begin{align*}
\varphi & = c_p \theta_0 \left[ \left( \frac{R_d}{\rho_0} \varrho \theta (1 + q_v/\epsilon) \right)^{R_d/c_v} - \pi_a \right] \\
\frac{d \varphi}{dt} & = \xi \varphi \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \varphi_a + \xi \varphi \left( \frac{\mathcal{R}_\theta}{\theta} + \frac{\mathcal{R}_{q_v}/\epsilon}{1 + q_v/\epsilon} \right) \\
\mathcal{R}_\theta & \equiv \text{RHS}(1c) + \mathbf{u} \cdot \nabla \theta_a
\end{align*}
\]
\[
\begin{aligned}
q_t &:= q_v + q_c + q_p \implies \frac{dq_t}{dt} = -\frac{1}{\varrho} \nabla \cdot (\varrho \mathbf{u} q_p) + D^{q_t} \\
q_l &:= q_c + q_p \implies \frac{dq_l}{dt} = (C_d + E_p) - \frac{1}{\varrho} \nabla \cdot (\varrho \mathbf{u} q_p) + D^{q_l} \\
q_t &\equiv 0 \implies q_v = q_c = q_p \equiv 0 \quad \text{recovers dry dynamics}
\end{aligned}
\]

\[
\begin{aligned}
\frac{\theta(1 + q_v/\varepsilon)}{\theta_0(1 + q_t)} = \frac{\Theta}{\Theta_0} &:= \Theta_q \implies g(1 - \Theta_q/\Theta_{qa}) \approx \frac{g}{\Theta_a} \left( \Theta' + \Theta_a(\varepsilon q'_v - q_c - q_p) \right) \\
\frac{\theta}{\Theta_{qa}} \approx \frac{\theta}{\Theta_a} &\implies \Theta' = \theta - \theta_a \quad \text{while} \quad q'_v = q_v - \frac{\theta}{\Theta_a} q_{va} \\
0 = -c_p \theta_{qa} \mathbf{G} \nabla \pi_a - g - \mathbf{f} \times \mathbf{u}_a + \mathcal{M}(\mathbf{u}_a)
\end{aligned}
\]

\[
\begin{aligned}
&q_c > 0 \implies q_v = q_{vs} \\
&q_v < q_{vs} \implies q_c = 0 \\
q_{vs} &= \frac{\varepsilon e_s}{p - e_s} \\
e_s(T) &= e_o \exp \left[ \frac{L}{R_v} \left( \frac{1}{T_o} - \frac{1}{T} \right) \right]
\end{aligned}
\]
Governing equations, actually solved conservation form:

\[
\frac{\partial \mathcal{G}_q}{\partial t} + \nabla \cdot (\mathcal{G}_q \mathbf{v}) = 0 ,
\]

\[
\frac{\partial \mathcal{G}_q \mathbf{u}}{\partial t} + \nabla \cdot (\mathcal{G}_q \mathbf{v} \otimes \mathbf{u}) = \mathcal{G}_q \left( - \Theta_e \tilde{\mathbf{G}} \nabla \varphi - \frac{g}{\Theta_a} (\theta' + \Theta_a (\varepsilon q'_v - q_c - q_p)) - \mathbf{f} \times \mathbf{u}' + \mathbf{M}' + \mathbf{D} \right) ,
\]

\[
\frac{\partial \mathcal{G}_q \theta'}{\partial t} + \nabla \cdot (\mathcal{G}_q \mathbf{v} \theta') = \mathcal{G}_q \left( - \tilde{\mathbf{G}}^T \mathbf{u} \cdot \nabla \Theta_a + \frac{L \theta}{c_p T} (C_d + E_p) + \mathcal{H} \right) ,
\]

\[
\frac{\partial \mathcal{G}_q q_{\ldots}}{\partial t} + \nabla \cdot (\mathcal{G}_q \mathbf{v} q_{\ldots}) = \mathcal{G}_q \left( \mathcal{R}^{q_{\ldots}} \right) .
\]

**bulk microphysics**

\[
\frac{\partial \mathcal{G}_q \varphi}{\partial t} + \nabla \cdot (\mathcal{G}_q \mathbf{v} \varphi) = \mathcal{G}_q \mathcal{R}^\varphi
\]

\[
\mathcal{R}^\varphi = -\xi \phi \frac{1}{\mathcal{G}} \nabla \cdot (\mathcal{G} \mathbf{v}) - \frac{1}{\mathcal{G}_q} \nabla \cdot (\mathcal{G}_q \mathbf{v} \phi_a) + \phi_a \frac{1}{\mathcal{G}_q} \nabla \cdot (\mathcal{G}_q \mathbf{v}) + \xi \phi \Pi
\]

\[
\Pi \equiv \left( \frac{\mathcal{R}^\theta}{\theta} + \frac{\mathcal{R}^{q_v}/\varepsilon}{1 + q_v/\varepsilon} \right)
\]

**conservativeness, relative simplicity, Eulerian/Lagrangian congruence**
\[
\frac{d\Phi}{dt} = \mathcal{R} \equiv \mathcal{R}_c + \mathcal{R}_p
\]

\[
\Phi = \begin{bmatrix} \theta \\ q_v \\ q_c \\ q_p \end{bmatrix} \quad \mathcal{R}_c = \begin{bmatrix} \frac{L\theta}{c_p T} C_d \\ -C_d \\ C_d \\ 0 \end{bmatrix} \quad \mathcal{R}_p = \begin{bmatrix} \frac{L\theta}{c_p T} E_p \\ -E_p \\ -A_p - C_p \\ A_p + C_p + E_p + F_p \end{bmatrix}
\]

\[
F_p := -\frac{1}{q} \nabla \cdot (q u \pm q_p)
\]

\[
\Phi(x,t) = \Phi(x_0, t_0) + \int_T (\mathcal{R}_c + \mathcal{R}_p) \, dt \quad \Rightarrow \quad \Phi^{n+1} = \left( \Phi + 0.5 \delta t \mathcal{R}_c \right)_o + 0.5 \delta t \mathcal{R}_c^{n+1} + \delta t \langle \mathcal{R}_p \rangle^{n+1}_n
\]

e.g. \( (A_p + C_p)^{n+1} = -\mathcal{R}_p^{q_c}|^{n+1} \quad \text{and} \quad E_{p}^{n+1} = -\mathcal{R}_p^{q_v}|^{n+1} \quad \Rightarrow \quad \mathcal{O}(\delta t) \) estimate

\[
\tilde{q}_p^{n+1} = (q_p + \delta t \mathcal{R}_p^{q_p})_o - \delta t (\tilde{R}_p^{q_c} + \tilde{R}_p^{q_v})^{n+1}
\]

\[
(q_c + \delta t \mathcal{R}_p^{q_c})_o + \delta t \mathcal{R}_p^{q_c}|^{n+1} \geq 0 \quad \Rightarrow \quad -\mathcal{R}_p^{q_c}|^{n+1} = \min \left( \mathcal{R}_p^{q_c}|^{n+1}, (\delta t^{-1} q_c + \mathcal{R}_p^{q_c})_o \right),
\]

\[
(q_p + \delta t \mathcal{R}_p^{q_p})_o - \delta t (\tilde{R}_p^{q_c}|^{n+1} + \mathcal{R}_p^{q_v}|^{n+1}) \geq 0 \quad \Rightarrow \quad -\mathcal{R}_p^{q_v}|^{n+1} = \max \left( \mathcal{R}_p^{q_v}|^{n+1}, -\left(\delta t^{-1} q_p + \mathcal{R}_p^{q_v}\right)_o + \tilde{R}_p^{q_c}|^{n+1} \right)
\]
Conservative FV formulation, positivity of advection, and EU/SL congruence enable consistent semi-implicit integration conserving water substance to machine precision.

\[
\tilde{q}_p = \mathcal{L} \left( \tilde{q}_p^{n+1}, G^T u^\dagger, (G q)^{n+1}, \delta t \right)
\]

\[
R_p^{\tilde{q}_p} |^{n+1} = (\mathcal{R}_p^{\tilde{q}_c} + \mathcal{R}_p^{\tilde{q}_v})^{n+1} + (\tilde{q}_p - \tilde{q}_p^{n+1}) / \delta t
\]

Completed all explicit integrals \( \Rightarrow \) the implicit integrals follow

\[
q_v^{n+1} = \tilde{q}_v - \Delta q_c = q_{vs} \left( \hat{\theta} + \Lambda^* \Delta q_c, \Lambda^* \right), \text{ if } (\hat{q}_c + \Delta q_c) > 0
\]

\[
q_c^{n+1} = \hat{q}_c + \Delta q_c = 0, \text{ if } (\hat{q}_v - \Delta q_c) < q_{vs} \left( \hat{\theta} + \Lambda^* \Delta q_c, \Lambda^* \right)
\]

Saturation condition:

\[
\Delta q_c = \frac{dq_{vs}}{d\Delta q_c} \Delta q_c \bigg|_{\Delta q_c=0} \Delta q_c
\]

\[
\Delta q_c = \left[ \tilde{q}_v - q_{vs} \left( \hat{\theta}, \Lambda^* \right) \right] \left[ 1 + \frac{dq_{vs}}{d\Delta q_c} \bigg|_{\Delta q_c=0} \right]^{-1}
\]

\[
\hat{\theta}^+ = \hat{\theta} + \Lambda^* \Delta q_c , \quad \tilde{q}_v^+ = \tilde{q}_v - \Delta q_c
\]

\[
\Delta^+ q_c = \Delta q_c + \text{EQS}_{(38)} \left( \tilde{q}_v^+, \hat{\theta}^+ \right)
\]
Flow chart for the whole system:

\[
\begin{align*}
\varrho_i^{n+1} &= A_i\left(\varrho^n, (\mathcal{G}\mathbf{v})^{n+1/2}, \mathcal{G}^n, \mathcal{G}^{n+1}\right) \implies \mathbf{V}^{n+1/2} = (\mathcal{G}\varrho\mathbf{v})^{n+1/2} \\
\hat{\mathbf{u}}_i &= A_i\left(\hat{\mathbf{u}}, \mathbf{V}^{n+1/2}, \varrho^*, n, \varrho^{n+1}\right) \\
\hat{\theta}_i &= A_i\left(\hat{\theta}, \mathbf{V}^{n+1/2}, \varrho^*, n, \varrho^{n+1}\right) \quad \text{and} \quad \hat{R}^0_i = A_i\left(\mathcal{R}^0|n + \mathcal{H}(q)|n, \mathbf{V}^{n+1/2}, \varrho^*, n, \varrho^{n+1}\right), \\
\tilde{\mathcal{G}}..._i &= A_i\left(\tilde{\mathcal{G}}..._n, \mathbf{V}^{n+1/2}, \varrho^*, n, \varrho^{n+1}\right) \\
\mathcal{P}\mathcal{C}\mathcal{M}\left(\hat{\theta}, \tilde{\mathcal{G}}..._n, \hat{R}^0, \hat{R}^q...\right) &\implies \left[\theta, q..._n, \mathcal{R}^0, \mathcal{R}^q...\right]^{n+1} \quad \text{and} \quad \mathbf{B}^q := \theta_a(\varepsilon q_v' - q_c - q_d)
\end{align*}
\]

\[
\begin{align*}
\theta_i' &= \hat{\theta}_i - 0.5\delta t \left(\Theta^T \mathbf{u} \cdot \nabla\theta_a\right)_i \\
u_i &= \hat{\mathbf{u}}_i - 0.5\delta t \left(\Theta^* \tilde{\mathbf{G}} \nabla \varphi + \frac{\varrho}{\theta_a} [\theta' + \mathcal{B}^{q,n+1}]\right)_i \\
&\quad - 0.5\delta t \left(\mathbf{f} \times [\mathbf{u} - \frac{\theta^*}{\theta_a} \mathbf{u}_a] - \mathcal{M}'(\mathbf{u}^*, \mathbf{u}_a, \frac{\theta^*}{\theta_a})\right)_i \\
\mathbf{L} \mathbf{u} &= \hat{\mathbf{u}} - 0.5\delta t \Theta^* \tilde{\mathbf{G}} \nabla \varphi \implies \mathbf{u} = \hat{\mathbf{u}} - \mathbf{C} \nabla \varphi \implies \mathbf{v} = \hat{\mathbf{v}} - \tilde{\mathbf{G}}^T \mathbf{C} \nabla \varphi \implies \\
0 &= -\sum_{\ell=1}^3 \left(\frac{\mathbf{A}^*_{\ell}}{\zeta_{\ell}} \nabla \cdot \zeta_{\ell}(\hat{\mathbf{v}} - \tilde{\mathbf{G}}^T \mathbf{C} \nabla \varphi)\right) = \mathbf{B}^*(\varphi - \hat{\varphi})
\end{align*}
\]
Figure 11: Horizontal cross-sections for the simulation with $X = 120$ and $A = 500$ m at 5 km at 30 min intervals for (a) vertical velocity ($c_1 = 2$ m s$^{-1}$) and (b) rainwater ($c_2 = 1$ g kg$^{-1}$). Here the longitudinal positions are displayed in the ground-relative framework.
conservative FV formulation + positivity of advection + Eulerian/Lagrangian congruence enable effective integrations of all-scale PDEs of moist-precipitating atmospheric dynamics, while conserving $H_2O$ substance to round-off error.

Fig. 3. FVM results for the tropical cyclone experiment of DCMIP (experiment-id: 162) after 10 days of simulation. Horizontal wind magnitude [m/s] in the horizontal (left) and zonal-height cross sections. Longitude [deg] marks abscissas of both panels, whereas latitude [deg] and altitude [km], marks ordinates of, respectively, right and left panel.

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Further reading:

The modelling infrastructure of the Integrated Forecasting System: Recent advances and future challenges


Research Department

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Special topic paper presented at the 44th session of ECMWF’s Scientific Advisory Committee, Reading, UK

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A finite-volume module for simulating global all-scale atmospheric flows

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ABSTRACT

The paper documents the development of a global nonhydrostatic finite-volume module designed to enhance an established spectral-transform based numerical weather prediction (NWP) model. The module adheres to NWP standards with formulation of the governing equations based on the classical meteorological latitude-longitude spherical framework. In the horizontal, a bespoke unstructured mesh with finite-volumes built about the reduced Gaussian grid of the existing NWP model comprises the numerous singularities in the polar regions of the spherical framework. All dependent variables are co-located, accommodating both spectral-transform and grid-point solutions at the same physical locations. In the vertical, a uniform finite-difference discretisation facilitates the solution of incompressible elliptic problems in thin spherical shells, while the pliability of the physical vertical coordinate is delegated to generalised continuous transformations between computational and physical space. The newly developed module assumes the compatible linear equations as default, but includes reduced soundproof PDEs as an option. Furthermore, it employs semi-implicit forward-in-time integrators of the governing PDEs system, akin to but more general than those used in the NWP model. The module shares the equal region parallelisation scheme with the NWP model, with multiple layers of parallelisation hybridising MPI ranks and OpenMP threads. The efficacy of the developed nonhydrostatic module is illustrated with benchmarks of idealised global weather.

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1. Introduction

Numerical weather prediction (NWP) has achieved high proficiency over the past 30 years. This owes much to advances in computer hardware, observational networks and data assimilation techniques as well as numerical methods for integrating hydrostatic primitive equations (HPE). One particular numerical approach embraced widely by NWP combines semi-implicit time stepping with semi-Lagrangian advection (SLA) and with spectral-transform spatial discretisation of the governing HPE [10]. The SLA time stepping enables integrations with Courant numbers of the fluid flow and wave motions much larger than unity, whereas the spectral-transform discretisation facilitates the efficient solution of elliptic equations induced by the SLA approach. Moreover, it circumvents the computational expense of the latitude-longitude (lat-lon) co-