Computational Modes in Weather and Climate Models?

John Thuburn (University of Exeter, UK)

With thanks to Colin Cotter, Dan Holdaway, Tom Melvin, Daniel Le Roux, David Long, Andrew Staniforth, Mark Taylor, Hilary Weller

ECMWF seminar, 2-5 September 2013

Introduction

Wavelike solutions of the discrete equations that cannot (easily) be identified with a corresponding solution of the continuous equations are often called **'computational modes'**

Often revealed by von Neumann type normal mode analysis (but laborious for any but the simplest schemes), or (for stationary modes) by kernel analysis.

Why are they bad?

Even if the initial conditions have zero amplitude computational modes, they will certainly be excited by nonlinearity, physical parametrizations, data assimilation.

They can adversely affect convergence.

They are often characterized by small spatial and/or temporal scales.

They may manifest themselves as a noisy solution, a failure to adjust correctly towards balance, a spurious release of instability, or an incorrect response to forcing.

It may be possible to filter them, but even then we are wasting DoFs.

Outline

- Introduction
- Classical examples: pressure modes, velocity modes, the Lorenz grid computational mode
- Parasitic modes
- Temporal computational modes, and space-time interaction
- Families of computational modes: triangular C-grid, hexagonal C-grid
- Trapped modes
- Summary

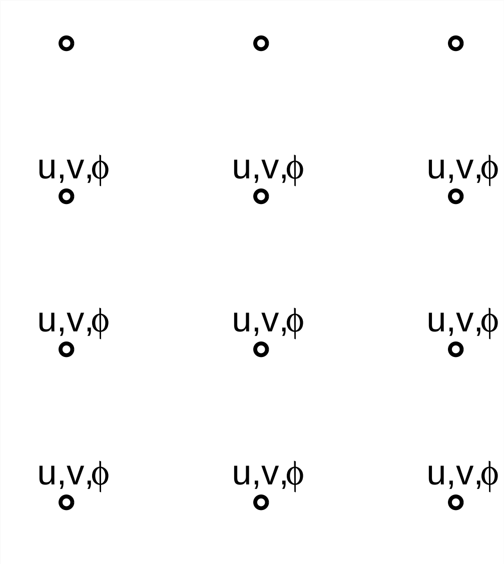
Classical examples: modes that fail to propagate

Typically, a particular spatial pattern is invisible to the dynamics, for example because of some averaging.

Pressure modes

Square A-grid:

$$
\begin{aligned}
\phi_t + \Phi(u_x + v_y) &= 0 \\
u_t - fv + \phi_x &= 0 \\
v_t + fu + \phi_y &= 0
\end{aligned}
$$

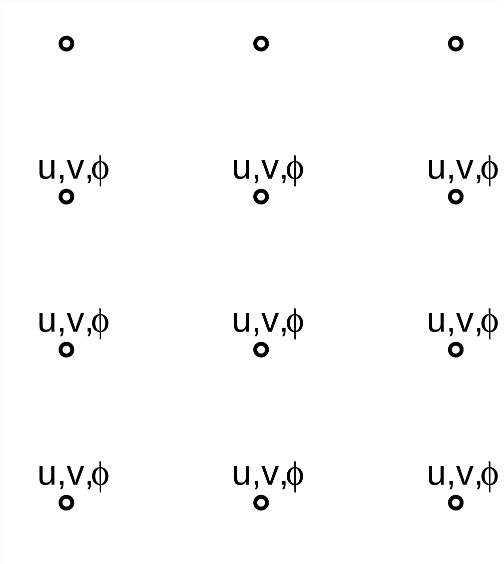


A checkerboard pattern in ϕ gives $\nabla\phi \approx 0$

Velocity modes

Square A-grid:

$$
\begin{aligned}
\phi_t + \Phi(u_x + v_y) &= 0 \\
u_t + \phi_x &= 0 \\
v_t + \phi_y &= 0
\end{aligned}
$$

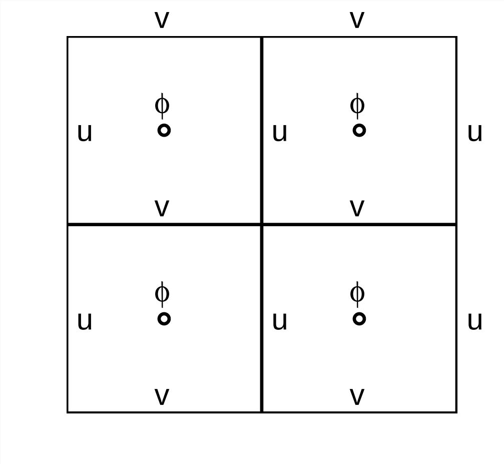


If $f = 0$, a checkerboard u and/or v pattern gives $u_x + v_y \approx 0$.

C-grid Coriolis mode

Square C-grid:

$$\begin{aligned} \phi_t + \Phi(u_x + v_y) &= 0 \\ u_t - fv + \phi_x &= 0 \\ v_t + fu + \phi_y &= 0 \end{aligned}$$



A certain checkerboard pattern in u and v has $u_x + v_y \approx 0$ and $fv = fu \approx 0$.

In FEM literature such modes are called CD modes because they are in the kernel of the C and D operators.

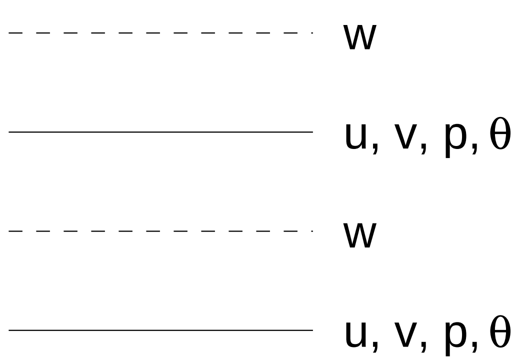
Lorenz grid computational mode

E.g.

$$c_p\theta\Pi_z + g = 0$$

where $\Pi = (p/p_0)^\kappa$.

Discrete solution is not unique.



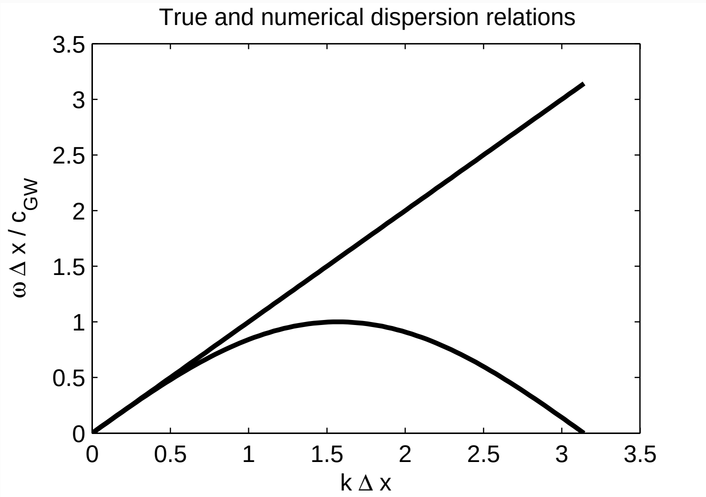
Schneider (MWR 1987): wrong response to steady forcing.

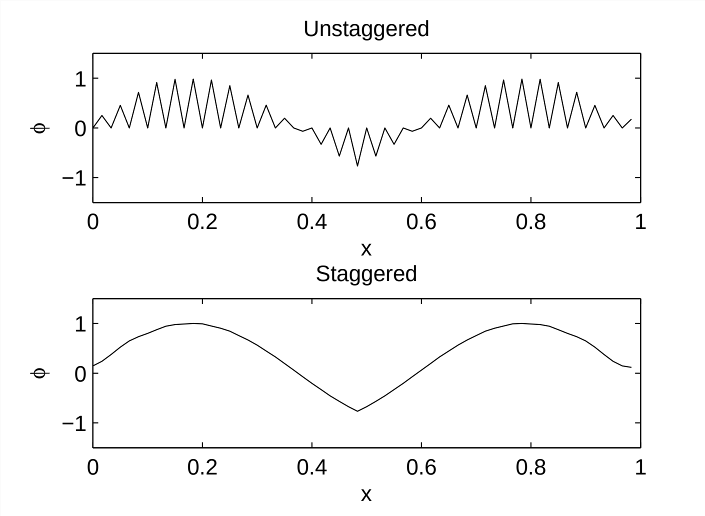
Arakawa and Moorthi (JAS 1988): spurious release of baroclinic instability.

Parasitic modes

All of the above are just the finest resolvable mode at the end of a spectrum of badly behaved modes.

E.g. $\phi_t + \Phi u_x = 0;$ $u_t + \phi_x = 0;$ 1D unstaggered grid





Group velocity of the wrong sign; incorrect response to forcing.

Temporal computational modes

Typically occur for schemes that use more than two time time levels, e.g. leapfrog, Adams-Bashforth, and have short timescale $O(\Delta t)$.

If the true dispersion relation has N modes for each wavenumber then we get an extra N *families* of modes for every extra time level.

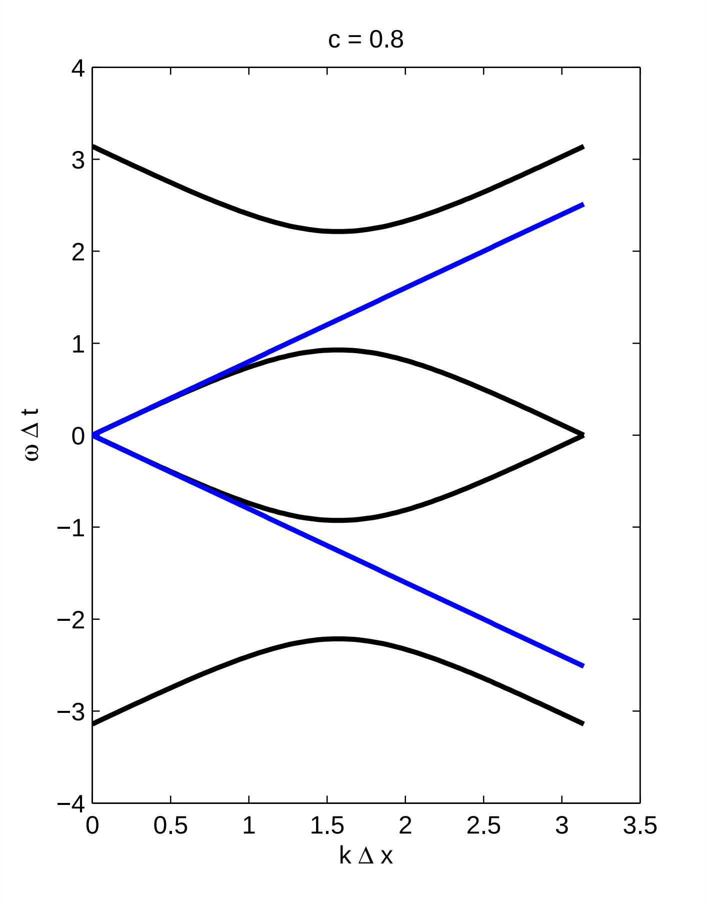
Modes whose amplification factor does not satisfy $A \to 1$ as $\Delta t \to 0$ are computational modes.

E.g.

CTCS for 1D nonrotating linear SWEs

$$\phi_t + \Phi u_x = 0$$
$$u_t + \phi_x = 0$$

unstaggered grid

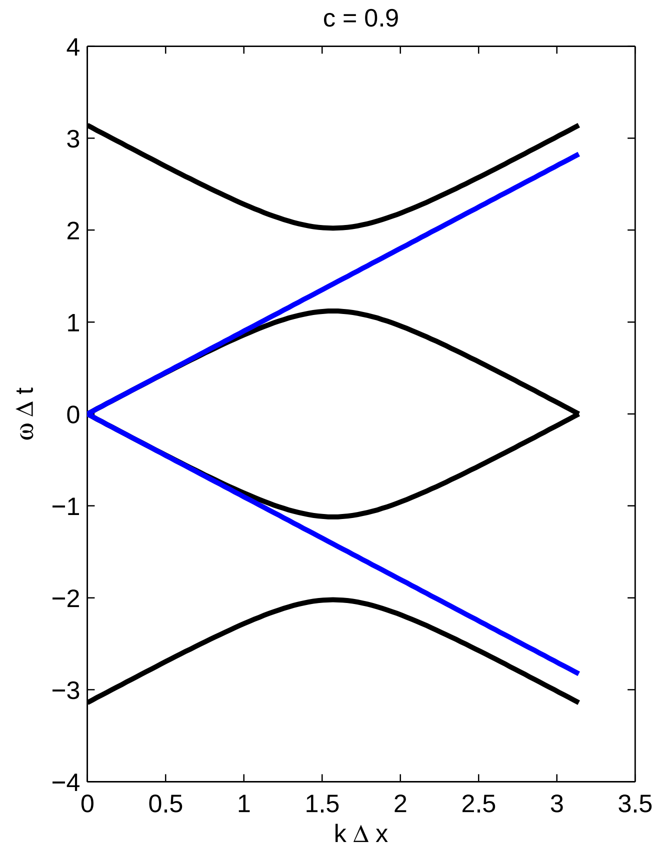


E.g.

CTCS for 1D nonrotating linear SWEs

$$\begin{aligned} \phi_t + \Phi u_x &= 0 \\ u_t + \phi_x &= 0 \end{aligned}$$

unstaggered grid

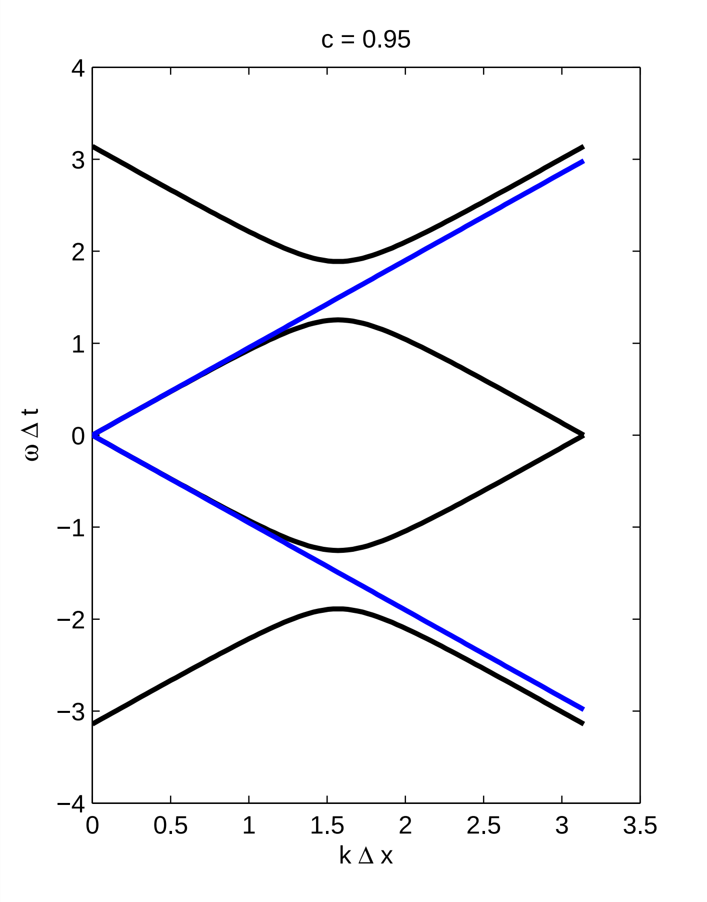


E.g.

CTCS for 1D nonrotating linear SWEs

$$\phi_t + \Phi u_x = 0$$
$$u_t + \phi_x = 0$$

unstaggered grid

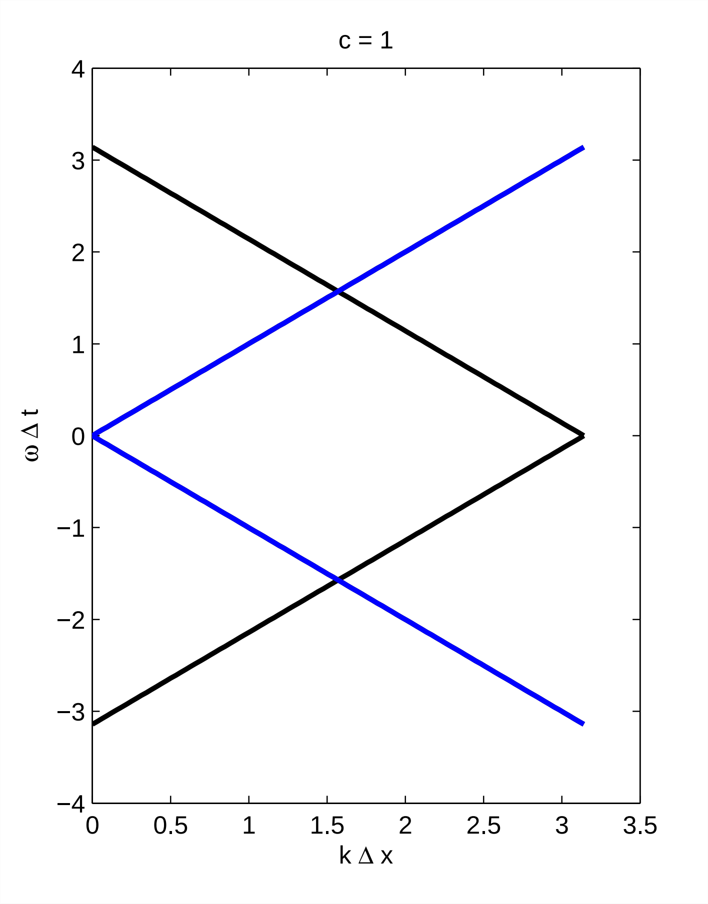


E.g.

CTCS for 1D nonrotating linear SWEs

$$\begin{aligned} \phi_t + \Phi u_x &= 0 \\ u_t + \phi_x &= 0 \end{aligned}$$

unstaggered grid

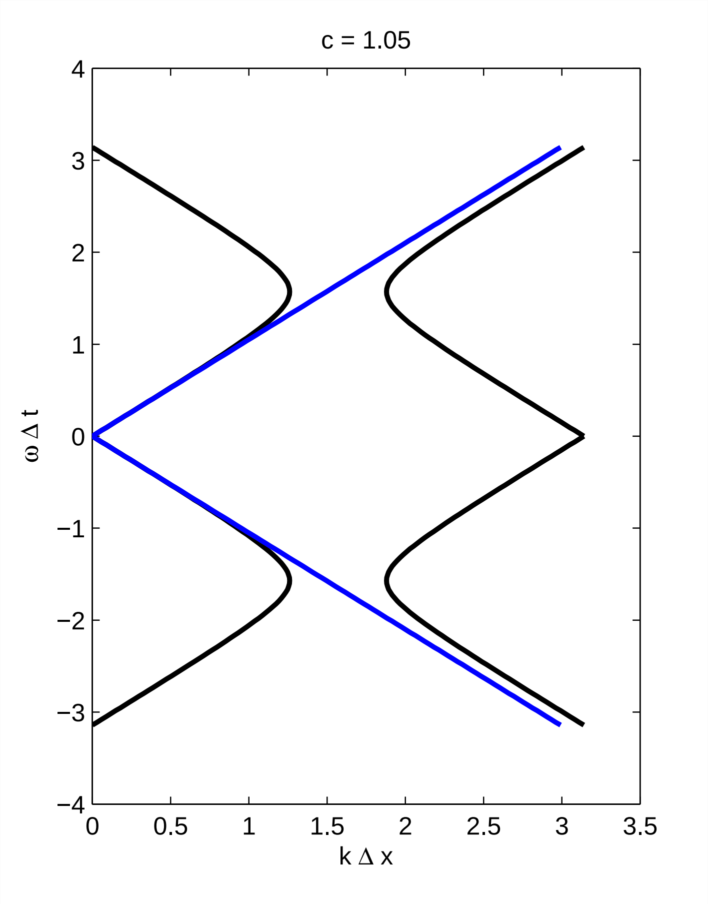


E.g.

CTCS for 1D nonrotating linear SWEs

$$\phi_t + \Phi u_x = 0$$
$$u_t + \phi_x = 0$$

unstaggered grid

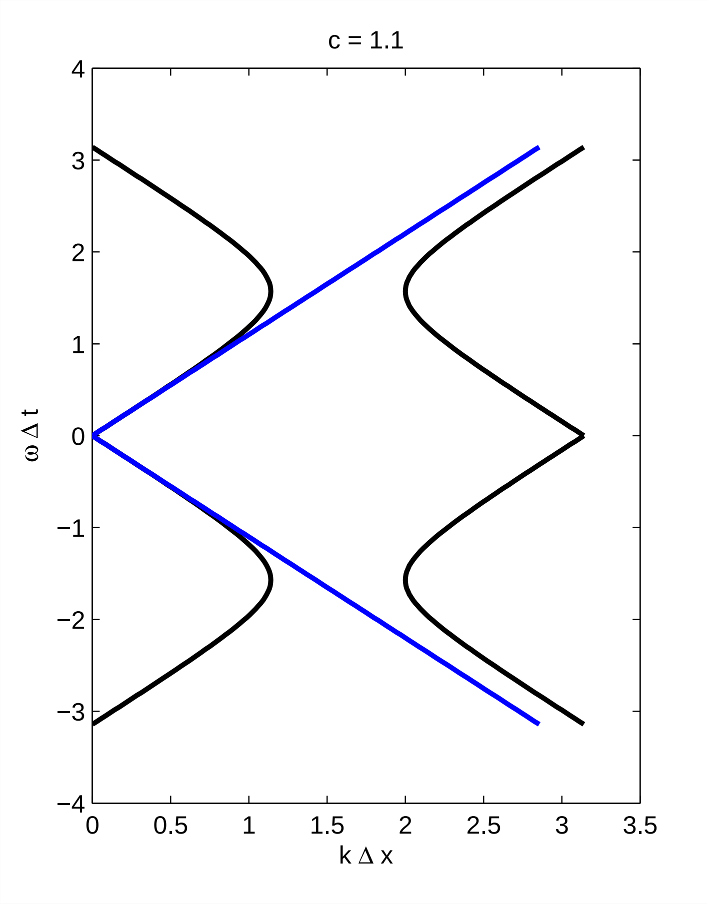


E.g.

CTCS for 1D nonrotating linear SWEs

$$\phi_t + \Phi u_x = 0$$
$$u_t + \phi_x = 0$$

unstaggered grid

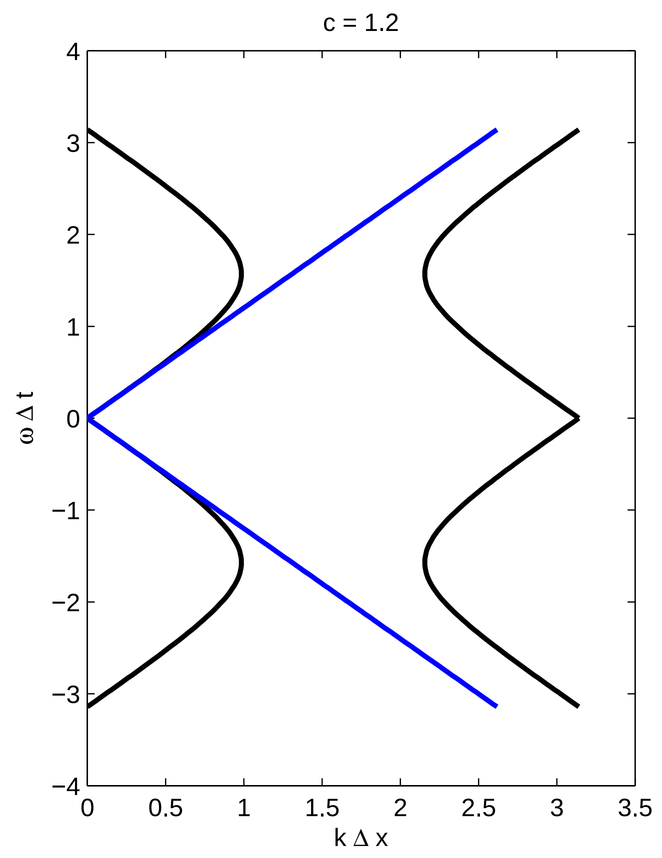


E.g.

CTCS for 1D nonrotating linear SWEs

$$
\begin{aligned}
\phi_t + \Phi u_x &= 0 \\
u_t + \phi_x &= 0
\end{aligned}
$$

unstaggered grid



Families of computational modes

For the rotating 2D shallow water equations we expect/hope to get a cubic dispersion relation for $\omega(\mathbf{k})$:

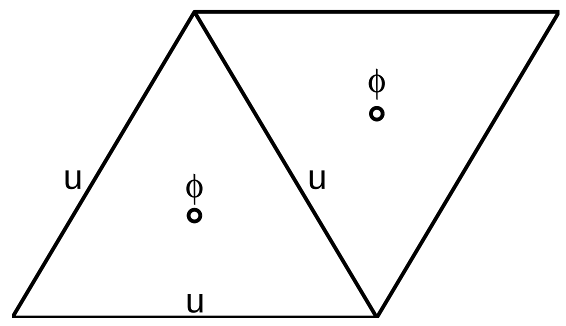
$$\omega(\omega^2 - f^2 - \Phi\mathbf{k}\cdot\mathbf{k}) = 0$$

Depending of the numbers of DoFs, we might get a higher degree polynomial.

Then what are the extra roots?

Example: the triangular C-grid

There are 5 DoF per basic repeating grid unit.
We get a quintic dispersion relation.
(Danilov, Ocean Dyn 2010)



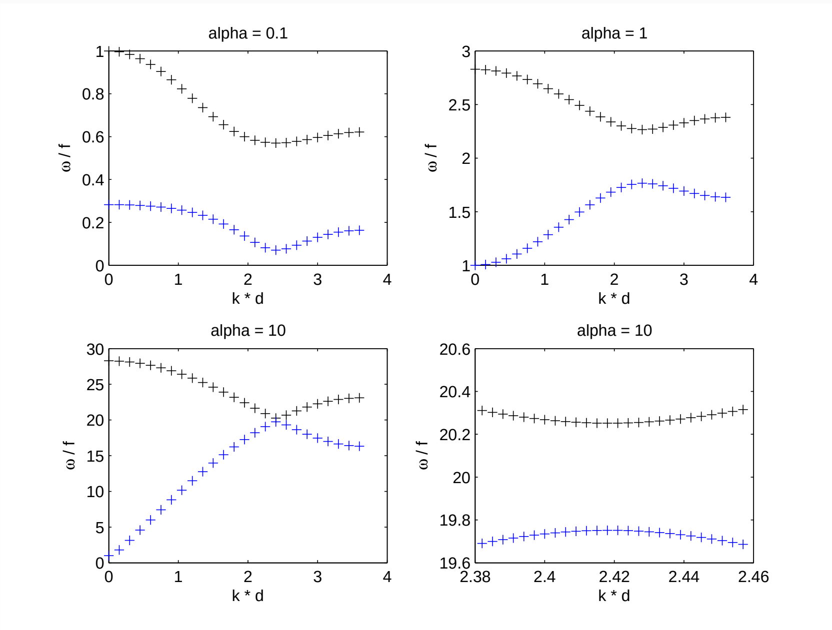
$$\frac{\omega}{f}\left\{\left(\frac{\omega}{f}\right)^4-\left(\frac{\omega}{f}\right)^2(3Q+C)+3Q^2(1-C)+Q\,(4C-1)\right\}=0$$

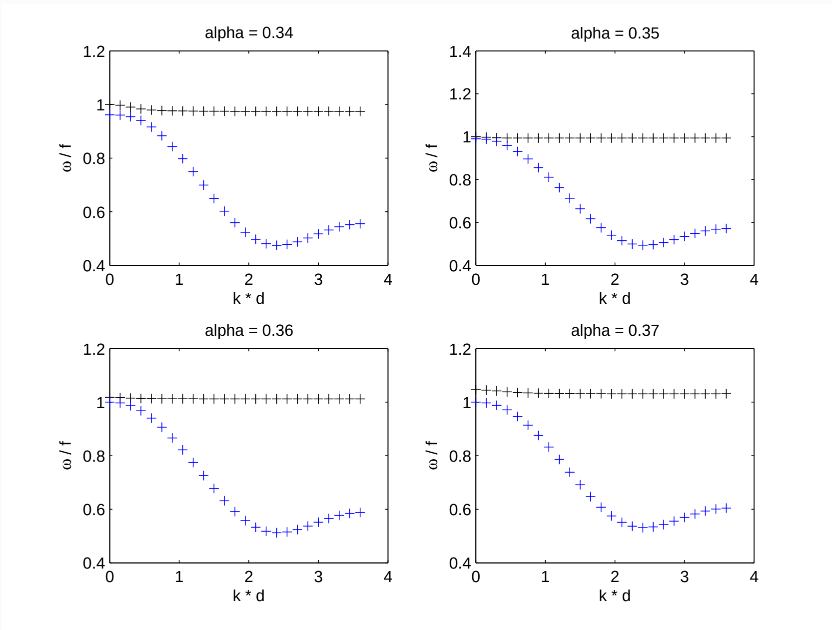
where $Q=8\alpha^2/3$. One geostrophic mode, two IGW modes, and two more IGW-like modes.

Rule of thumb: count pressure, divergence, and vorticity DoFs.

Which branch is the branch of computational modes?

$$\alpha = \text{Rossby radius}/\Delta$$

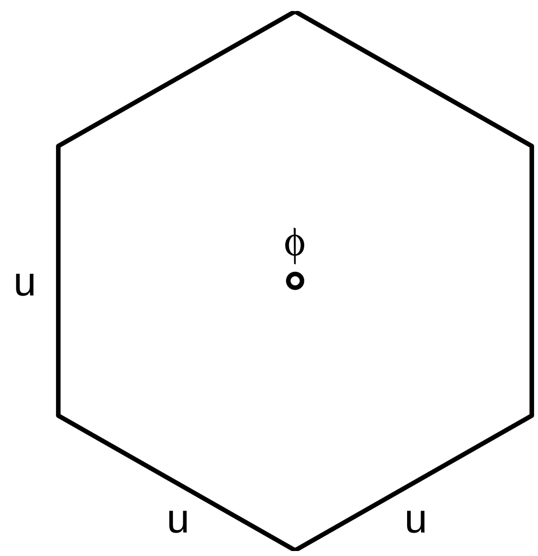




Similar behaviour for RT0 on triangles (Le Roux et al, SJSC 2007)

Example: hexagonal C-grid

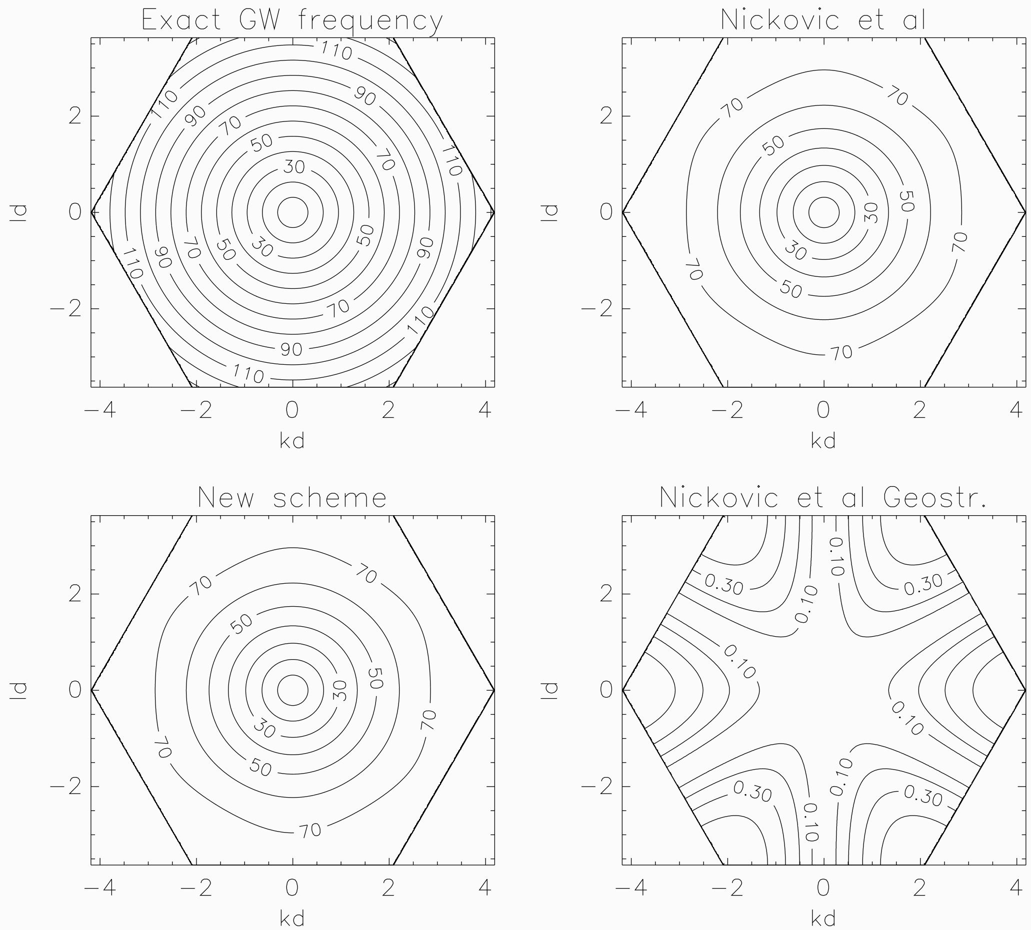
There are 4 DoF per basic repeating grid unit.
We get a quartic dispersion relation.
(Thuburn, JCP 2008)



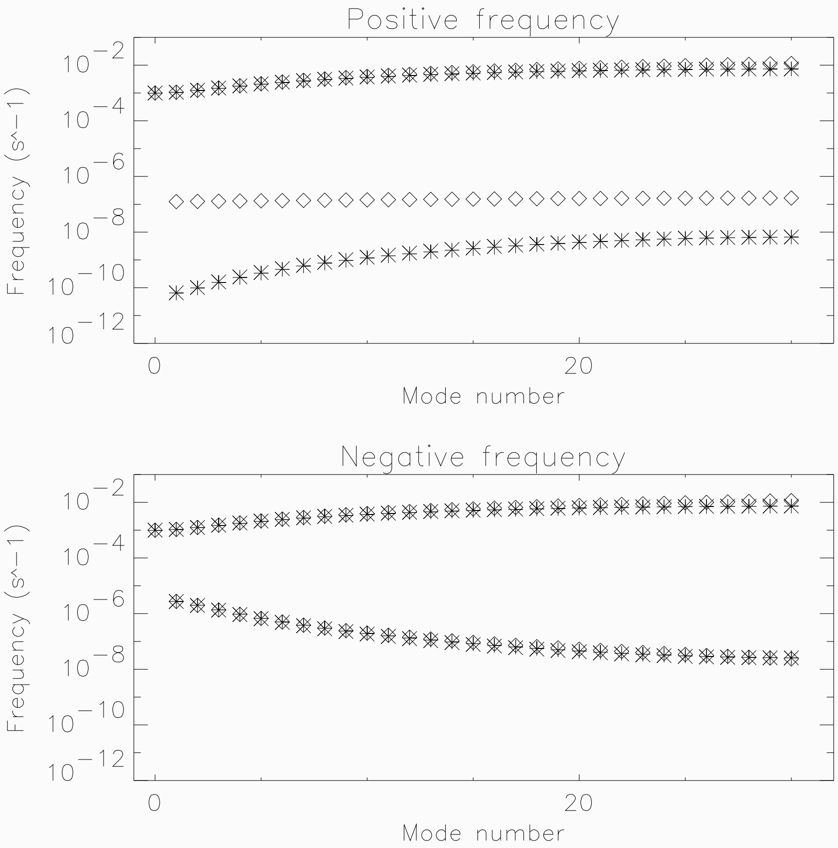
$$\left(\frac{\omega}{f}\right)^2 \left\{ \left(\frac{\omega}{f}\right)^2 - [T(\mathbf{k}) + QS(\mathbf{k})] \right\} = 0$$

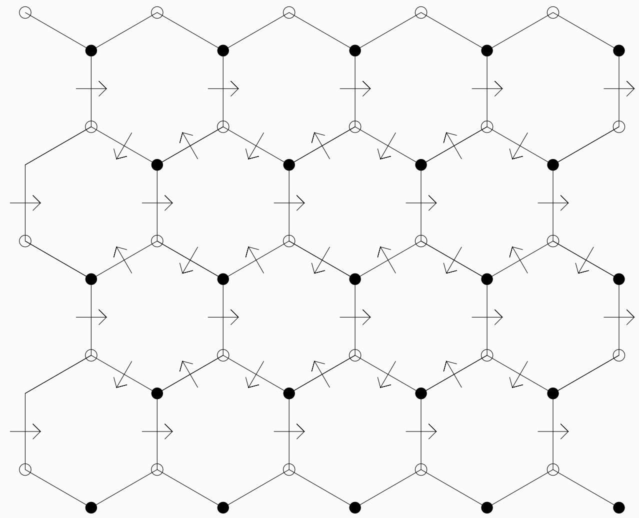
Two geostrophic modes and two IGW modes.

Aside: we must be careful in discretizing the Coriolis terms, otherwise we have **no** zero frequency geostrophic solutions (Ničković et al MWR 2002).



Is the second Rossby mode branch a branch of computational modes?





These modes are too passive

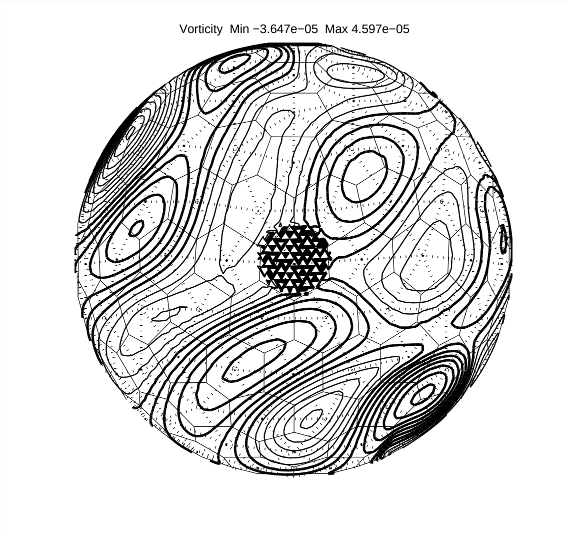
But...

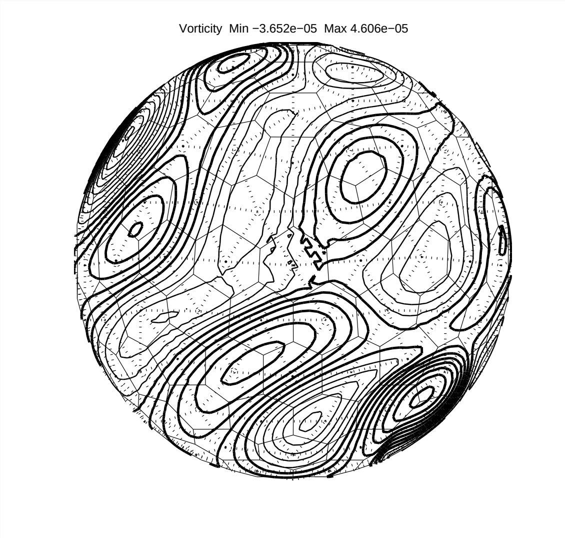
provided PV is accurately advected, they appear to be harmless because Doppler shifting rather than the Rossby wave mechanism dominates ω.

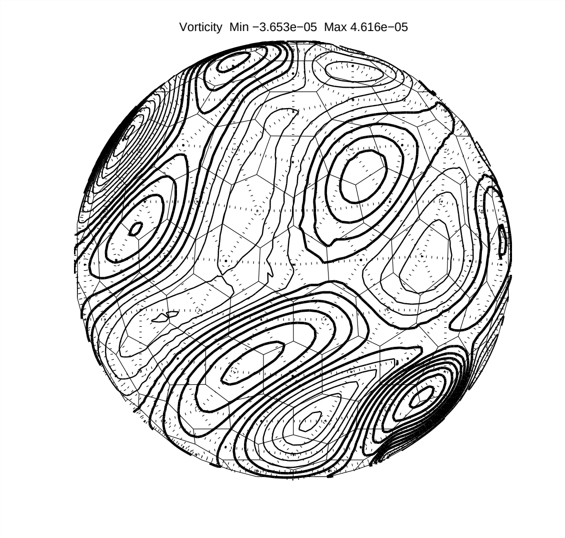
Also, scale-selective damping suppresses them.

Artificially introduced grid scale vorticity noise at day 15 of test case 5.

Hex grid, 10242 faces. Plots are 1 hour apart.

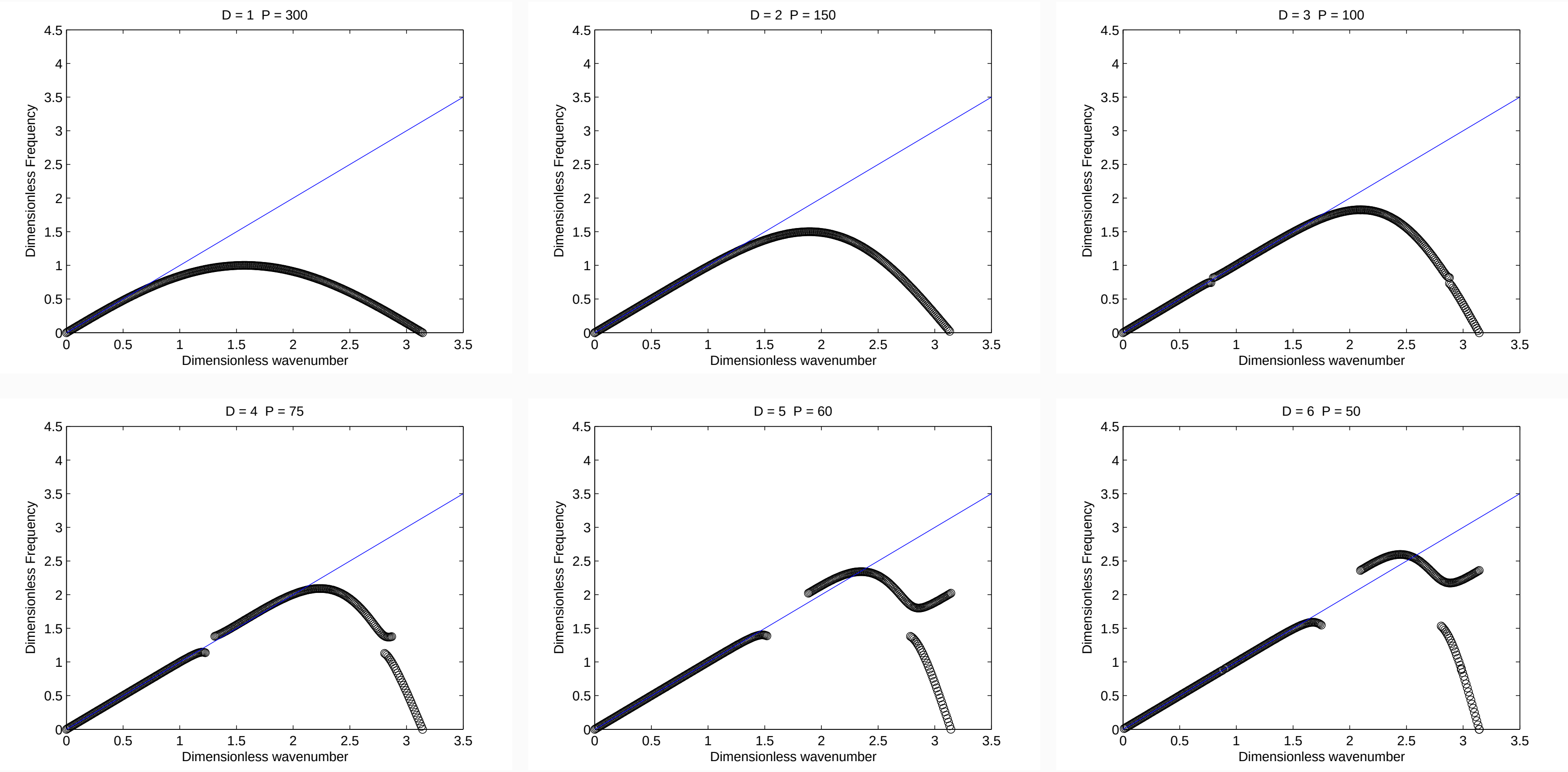






DoF per basic repeating unit is useful for Galerkin methods too

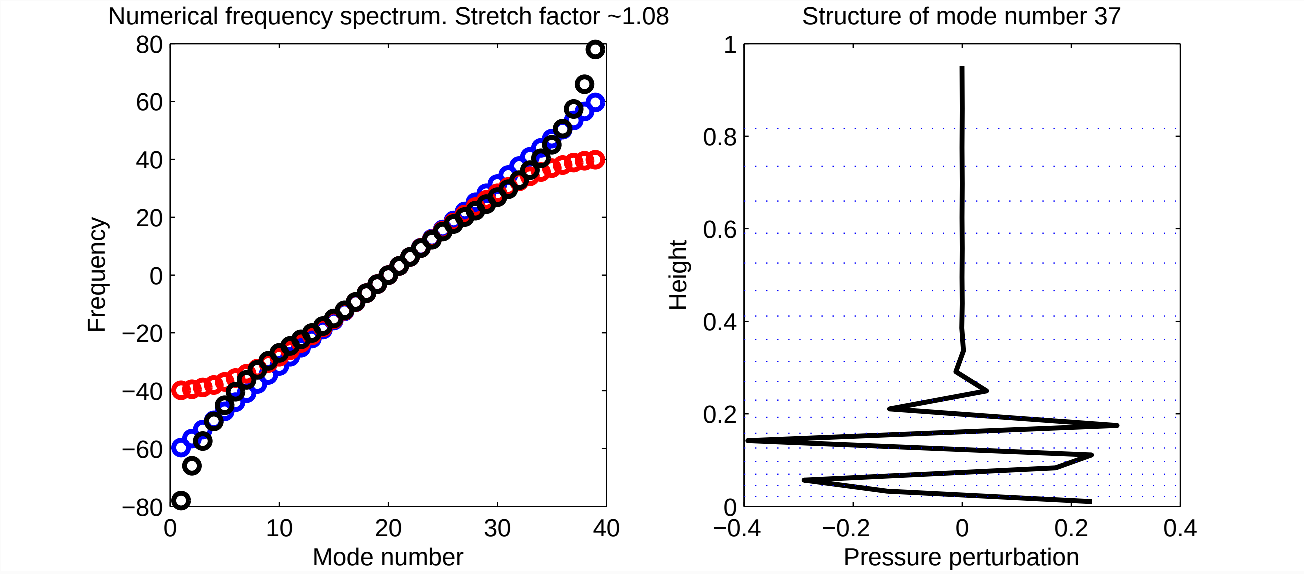
E.g. spectral element method, 1D nonrotating SWE (Melvin et al, QJRMS 2012)



There are many 2D finite element examples (Le Roux, JCP 2012). Modification to mass matrix, or disspation, can (sometimes) ‘glue’ the branches together.

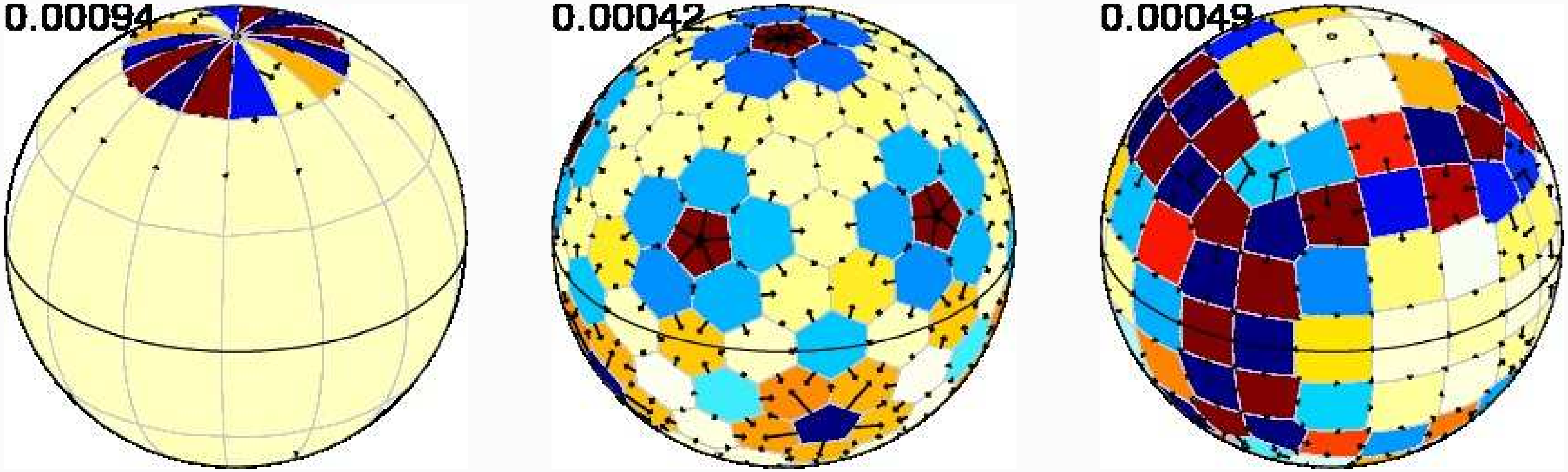
Trapped modes

$$p_t + c^2 w_z = 0$$
$$w_t + p_z = 0$$



Waves are reflected from the location where their group velocity goes to zero.

(Vichnevetsky, Appl Numer Math 1987, Long and Thuburn, JCP 2011) e.g. Highest frequency eigenmode of spherical rotating SWEs on lat-long, hexagonal Voronoi, and ‘Voronoi-ized’ cube grids



(Weller et al, MWR 2012)

Summary

• Numerical distortion of wave propagation takes a variety of forms;

it is hard to give a concise yet comprehensive definition of **'computational mode'**.

• Distorted wave propagation can damage numerical solutions in several ways (noise, convergence, adjustment, response to forcing...)

• Some 'computational modes' may be relatively harmless

(i) if the scheme may be modified so that extra branches become extensions of physical branches;

(ii) if the distorted physics is not dominant in determining the wave behaviour.