Horizontally-explicit vertically-implicit (HEVI) time-stepping methods for NWP and climate models

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Why bother?

[left:] Fig. 1(d) & [right:] Fig. 2(d), Ullrich & Jablonowski (2012, MWR),
courtesy Paul Ullrich
Outline

- Overview of the problem for time-stepping schemes
- Review of established HEVI methods
- Introduce ideas for a new HEVI approach
- Show some analyses of Runge-Kutta HEVI schemes:
  - linear analyses
  - numerical results
Time discretisation

Consider the atmospheric system as

$$\frac{\partial F}{\partial t} = g(F),$$  

(1)

where $F = [u, T, p]^T$.

For the discretised model, how do we use past solutions to approximate (1)?

LHS: \[ \frac{\partial F}{\partial t} \approx \frac{\phi^{n+1} - \phi^{n}}{\alpha \Delta t} \]

RHS: \[ g(F) \approx g\left(\phi^{?}\right) \]
Time discretisation

The atmospheric model is better described by

\[ \frac{\partial F}{\partial t} = f(F) + s(F), \]

where \( ||f|| \gg ||s||. \)

The difference in scales comes from 2 aspects:

1. **continuous model**: solutions comprise fast and slow modes, i.e. speed \((c)\);
2. **discretised model**: grid-spacings differ: \( \Delta x \gg \Delta z \), i.e. mesh.

The extent to which a time-stepping method can represent the model solution on a given mesh depends on \( \Delta t \).

Specifically, the *Courant* numbers

\[ c_x \frac{\Delta t}{\Delta x}, \quad c_z \frac{\Delta t}{\Delta z} \]

for a given model problem can be used to determine a discretisation method’s

- accuracy, and
- stability.
What are these fast and slow modes?

Many global weather models are based on nonhydrostatic, compressible equations.

From dispersion relation analyses, we can identify the major (dry) dynamical processes to be (from slowest to fastest):

- rotation
- advection \((U)\)
- gravity waves
- acoustic waves \((c_s)\)

such that \(c_s \gg U\),

\[
\Rightarrow c_s \frac{\Delta t}{\Delta x} \gg U \frac{\Delta t}{\Delta x}.
\]

⇒ Question becomes:

**How do we handle the acoustic waves?**
Approaches for handling the fast modes

Consider again the atmospheric model

\[
\frac{\partial F}{\partial t} = f(F) + s(F),
\]

where \( ||f|| \gg ||s|| \), due to

1. \( c_s \gg U \); and
2. \( \Delta x \gg \Delta z \).

We can:

- tackle (1) with filtered equations (R. Klein, D. Holm & P. Smolarkiewicz talks);
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- tackle (1) with filtered equations (R. Klein, D. Holm & P. Smolarkiewicz talks);

- tackle (1) & (2) with
  “semi-implicit” solutions (e.g. Tapp & White, 1976; Cullen, 1990)
  ⇒ no stability limit on $\Delta t$ for implicit solutions of fast modes
  ⇒ computationally expensive 3D Helmholtz problem to solve
  (N. Wood and P. Benard talks);
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- tackle (1) with filtered equations (R. Klein, D. Holm & P. Smolarkiewicz talks);
- tackle (1) & (2) with "semi-implicit" solutions (e.g. Tapp & White, 1976; Cullen, 1990)
  ⇒ no stability limit on \( \Delta t \) for implicit solutions of fast modes
  ⇒ computationally expensive 3D Helmholtz problem to solve (N. Wood and P. Benard talks);
- tackle (2) & accept horizontal constraint from (1) with a "HEVI" (horizontally-explicit vertically-implicit) approach.
HEVI approaches: “Split-explicit” time-stepping

- **vertical**: implicit (trapezoidal) for fast modes ⇒ no stability limit on $\Delta t$; 1D (column) ⇒ tridiagonal problem ⇒ computationally cheap & no implications for parallelisation;

- **horizontal**: choose $\Delta t$ appropriate for physically important modes; use **sub-steps** ($\Delta \tau = \Delta t/M$) to solve the fast modes.

Now, the atmospheric model is described by

$$ \frac{\partial F}{\partial t} = f_V (F') + f_H (F') + s (F) $$

with fast contributions acting in the vertical ($f_V$) and horizontal ($f_H$).
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Efficiency comes from:

- $s(\phi)$: costly, but only once per $\Delta t$;
- $f(\phi)$: multiple computations, but cheap
HEVI approaches: Split-explicit time-stepping

Approach is well-established and widely used, e.g.:

- Wicker & Skamarock (2002), Klemp et al. (2007) → WRF
- MPAS, NICAM ...
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Choice of scheme for the slow modes varies:

- **leapfrog:**
HEVI approaches: Split-explicit time-stepping

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- Wicker & Skamarock (2002), Klemp et al. (2007) → WRF
- MPAS, NICAM …

Choice of scheme for the slow modes varies:

- **3rd-order 3-stage Runge-Kutta:**

```
\[ \begin{align*}
&\phi^t \\
&\{s^t\}^{(1)} \\
&f_H^t \\
&\phi^{t+\Delta t/3} \\
&\phi^{t+2\Delta t/3} \\
&\phi^{t+\Delta t} \\
&f_H^{t+\Delta t} \\
&f_H^{t+2\Delta t} \\
&f_H^{t+3\Delta t} \\
&f_H^{t+4\Delta t} \\
&f_H^{t+5\Delta t} \\
\end{align*} \]
```
HEVI schemes: recent ideas

Recent work has proposed a simpler approach:

- **vertical**: implicit (trapezoidal) for fast modes $\Rightarrow$ no stability limit on $\Delta t$;
  - 1D (column) $\Rightarrow$ tridiagonal problem $\Rightarrow$ computationally cheap & no implications for parallelisation;

- **horizontal**: choose $\Delta t$ appropriate for (fastest) horizontal modes

Consider the atmospheric system to be described by:

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with $f \equiv f_V$ for fast contributions acting in the vertical and $s$ all other terms.
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$\Rightarrow \Delta t$ much smaller ($\approx \Delta \tau$ of split-explicit)

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Why?

- Concern that deep atmosphere models, O(100 km), cannot benefit from efficiency gains with the split-explicit approach since $U \approx c_s$ in stratospheric polar jet (Gassmann, 2012)

  $\Rightarrow \Delta t \rightarrow \Delta \tau \quad (M \rightarrow 1)$

- Split-explicit combination of schemes requires additional damping terms for stability (e.g. Baldauf, 2010)
New HEVI approaches

For a HEVI approach to solve

\[ \frac{\partial F}{\partial t} = f(F) + s(F), \]

we need an implicit scheme to solve terms in \( f \) and an explicit scheme for terms in \( s \), i.e. from wider literature:

“IMEX” (implicit-explicit) combination.

How do we select the “best” scheme (of many)?
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- Current literature: very little analysis for the atmospheric system.
- Some very recent analyses:
  - Giraldo et al. (2013): multi-step & multi-stage methods;
  - Ullrich & Jablonowski (2012): multi-stage (Runge-Kutta) methods;
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Runge-Kutta HEVI schemes

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\[ \text{Multi-step methods have computational modes:} \]
- can be inherently well-behaved (damped) (see e.g. Durran & Blossey, 2012);
- can require additional damping to control (e.g. leapfrog).
Runge-Kutta HEVI schemes

We focus on Runge-Kutta (single-step, multi-stage) methods

e.g. 3rd-order 3-stage Runge-Kutta:

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- can be inherently well-behaved (damped) (see e.g. Durran & Blossey, 2012);
- can require additional damping to control (e.g. leapfrog).

Multi-stage methods don’t support inherent computational modes.
Analyses of Runge-Kutta HEVI schemes

Our analyses include:

- identifying a number of Runge-Kutta (RK) IMEX schemes from the literature;
- linear analysis of a system supporting acoustic waves, considering errors in:
  - amplitude, and
  - phase;
- numerical experiments for a system supporting acoustic and gravity waves: considering errors and rates of convergence.
RK IMEX schemes

Consider the system

\[ y_t = s(y, t) + f(y, t), \]

with \( s \) slow terms and \( f \) fast terms.
RK IMEX schemes

The $\nu$-stage RK IMEX scheme that steps system

$$y_t = s(y, t) + f(y, t),$$

from time $t = n\Delta t$ to $t = (n+1)\Delta t$ is described by:

$$y^{(j)} = y^n + \Delta t \sum_{k=1}^{j-1} \tilde{a}_{jk} s\left(y^{(k)}, t + \tilde{c}_k \Delta t\right) + \Delta t \sum_{l=1}^{j} a_{jl} f\left(y^{(l)}, t + c_l \Delta t\right), \quad j = 1, \nu,$$

$$y^{n+1} = y^n + \Delta t \sum_{j=1}^{\nu} \tilde{\omega}_j s\left(y^{(j)}, t + \tilde{c}_j \Delta t\right) + \Delta t \sum_{j=1}^{\nu} \omega_j f\left(y^{(j)}, t + c_j \Delta t\right)$$
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or by the double Butcher tableau:

| $\tilde{c}_1$ | $\tilde{a}_{11}$ | $\cdots$ | $\tilde{a}_{1\nu}$ | $c_1$ | $a_{11}$ | $\cdots$ | $a_{1\nu}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\tilde{c}_\nu$ | $\tilde{a}_{\nu1}$ | $\cdots$ | $\tilde{a}_{\nu\nu}$ | $c_\nu$ | $a_{\nu1}$ | $\cdots$ | $a_{\nu\nu}$ |
| $\tilde{\omega}_1$ | $\cdots$ | $\tilde{\omega}_\nu$ | $\omega_1$ | $\cdots$ | $\omega_\nu$ |

where

$\tilde{a}_{jk} = 0, \ k \geq j, \quad a_{jk} = 0, \ k > j$; 

and

$$\tilde{c}_j = \sum_{k=1}^{j-1} \tilde{a}_{jk}, \quad c_j = \sum_{k=1}^{j} a_{jk};$$

$$\sum \tilde{\omega}_j = \sum \omega_j = 1.$$
RK IMEX schemes

From the double Butcher tableau:

\[
\begin{array}{c|ccc}
\tilde{c}_1 & \tilde{a}_{11} & \cdots & \tilde{a}_{1\nu} \\
\vdots & \vdots & & \vdots \\
\tilde{c}_\nu & \tilde{a}_{\nu 1} & \cdots & \tilde{a}_{\nu \nu} \\
\hline
\tilde{\omega}_1 & \cdots & \tilde{\omega}_\nu \\
\end{array}
\quad \begin{array}{c|ccc}
c_1 & a_{11} & \cdots & a_{1\nu} \\
\vdots & \vdots & & \vdots \\
c_\nu & a_{\nu 1} & \cdots & a_{\nu \nu} \\
\hline
\omega_1 & \cdots & \omega_\nu \\
\end{array},
\]

we can easily identify:

- order of accuracy (explicit / implicit / overall) (e.g. Pareschi & Russo, 2005)
- time-level “splitting”: none / partial / complete
- implicit final stage for balanced solution (e.g. Ascher et al., 1997)
- explicit tableau: strong-stability preserving condition (e.g. Spiteri & Ruuth, 2002)
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We use a naming convention (Pareschi & Russo, 2005):

\[ [\text{NAME}]k(s,\sigma,p) \]

for a \(k\)-order explicit scheme, with overall \(s\) implicit stages, \(\sigma\) explicit stages and \(p\)-order accuracy.
Linear analysis

We consider HEVI solution of a system of acoustic waves:

\[
\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} = 0
\]

\[
\frac{\partial w}{\partial t} + \frac{\partial P}{\partial z} = 0
\]

\[
\frac{\partial P}{\partial t} + c_s^2 \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) = 0
\]

explicit  implicit
Linear analysis

We consider HEVI solution of a system of acoustic waves:

\[
\begin{align*}
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\frac{\partial P}{\partial t} + c_s^2 \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) &= 0
\end{align*}
\]

"UFPreF":

\begin{itemize}
\item U-**Forward**, \textbf{U-Forward},
\item P-**Forward**, \textbf{P-Forward};
\end{itemize}
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"UFPreF": U-Forward, P-Forward;

but alternatively,

\[
\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} = 0
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\frac{\partial P}{\partial t} + c_s^2 \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) = 0
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"UFPreB": U-Forward, P-Backward*

*Allusion to “forward-backward” scheme (Mesinger, 1977)
Solutions to the acoustic system

We rewrite the system as

$$\mathbf{F}_t = -H_1 \mathbf{F}_x - H_2 \mathbf{F}_x - V \mathbf{F}_z,$$

(2)

where subscripts denote partial derivatives, and

$$\mathbf{F} = \begin{pmatrix} u \\ w \\ P \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_s^2 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & c_s^2 & 0 \end{pmatrix}. \quad$$
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Exact solutions to (2) are of the form

$$\mathbf{F}(x, z, t) = \mathbf{F}_0 e^{i(k_x x + k_z z - \omega t)}.$$

(3)
Solutions to the acoustic system

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Exact solutions to (2) are of the form

\[ \mathbf{F}(x, z, t) = \mathbf{F}_0 e^{i(k_x x + k_z z - \omega t)}. \]

Substituting (3) into (2) yields dispersion relation \( \omega = \pm c_s \sqrt{k_x^2 + k_z^2}, \) 0.

⇒ We know how the system truly amplifies between times \( t \) and \( t + \Delta t \):

\[ \mathbf{F}(t + \Delta t) = A_0 \mathbf{F}(t) \quad \Rightarrow \quad A_0 = e^{-i\omega \Delta t}, \]

which has neutral amplitude \( |A_0| = 1 \) and phase \( \theta_0 = -\omega \Delta t \).
Solutions to the acoustic system

We rewrite the system as

\[ \mathbf{F}_t = -H_1 \mathbf{F}_x - H_2 \mathbf{F}_x - V \mathbf{F}_z, \] (2)

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\[ \mathbf{F}(x, z, t) = \mathbf{F}_0 e^{i(k_x x + k_z z - \omega t)}. \] (3)

For analysing the time-stepping methods, we assume continuous spatial derivatives, i.e. from (3):

\[ \mathbf{F}_x = i k_x \mathbf{F}, \quad \mathbf{F}_z = i k_z \mathbf{F}. \]

Then (2) becomes

\[ \mathbf{F}_t = -i k_x H_1 \mathbf{F} - i k_x H_2 \mathbf{F} - i k_z V \mathbf{F}. \]
Numerical amplification factors for the acoustic system

Using a $\nu$-stage RK IMEX scheme to solve

$$\mathbf{F}_t = -i k_x H_1 \mathbf{F} - i k_x H_2 \mathbf{F} - i k_z V \mathbf{F},$$

we can define numerical amplification factors for the $j = 1: \nu$ sub-stages as

$$\mathbf{F}^{(j)} = \mathbf{A}^{(j)} \mathbf{F}^n$$

and for the final stage, from $t = n \Delta t$ to $t = (n+1) \Delta t$, as

$$\mathbf{F}^{n+1} = \mathbf{A} \mathbf{F}^n.$$

So, ...
Numerical amplification factors for the acoustic system

We construct amplification factors for a $\nu$-stage RK IMEX scheme for:

$$
A^{(j)} = I - \Delta t \sum_{k=1}^{j-1} \tilde{a}_{jk} i k x (H_1 + H_2) A^{(k)} - \Delta t \sum_{l=1}^{j} a_{jl} i k z V A^{(l)}, \quad j = 1, \nu;
$$

**UFPreF:**

$$
A = I - \Delta t \sum_{j=1}^{j-1} \tilde{w}_j i k x (H_1 + H_2) A^{(j)} - \Delta t \sum_{j=1}^{j} w_{j} i k z V A^{(j)},
$$

$$
A^{(j)} = I - \Delta t \sum_{k=1}^{j-1} \tilde{a}_{jk} i k x H_1 A^{(k)} - \Delta t \sum_{l=1}^{j} a_{jl} (i k x H_2 + i k z V) A^{(l)}, \quad j = 1, \nu;
$$

**UFPreB:**

$$
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$$

where $H_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_s^2 & 0 & 0 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & c_s^2 & 0 \end{pmatrix}$.
Numerical amplification factors for the acoustic system

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\mathbf{A}^{(j)} = \mathbf{I} - \Delta t \sum_{k=1}^{j-1} \tilde{a}_{jk} i k_x (H_1 + H_2) \mathbf{A}^{(k)} - \Delta t \sum_{l=1}^{\nu} a_{jl} i k_z V \mathbf{A}^{(l)}, \quad j = 1, \nu; \\
\text{UFPreF:}
$$

$$
\mathbf{A} = \mathbf{I} - \Delta t \sum_{j=1}^{j-1} \tilde{w}_j i k_x (H_1 + H_2) \mathbf{A}^{(j)} - \Delta t \sum_{j=1}^{\nu} w_j i k_z V \mathbf{A}^{(j)}, \\
$$

where $H_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_s^2 & 0 & 0 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & c_s^2 & 0 \end{pmatrix}$

Note: $\mathbf{A}$ is a $3 \times 3$ complex matrix — its eigenvalues ($\lambda_1, \lambda_2, \lambda_3$) describe the amplification factors of the three system modes: two acoustic and one non-divergent
Linear analyses

We numerically generate values for \( A \) (and \( \Rightarrow (\lambda_1, \lambda_2, \lambda_3) \)) and consider:

- amplitude errors: \( \Rightarrow \) instability?
- phase errors: \( \Rightarrow \) implied direction of group velocity?

We consider the acoustic Courant number (\( C_{s,x} \equiv c_s k_x \Delta t, \ C_{s,z} \equiv c_s k_z \Delta t \)) ranges:

- \( C_{s,x} \equiv \Delta t^* \in [0, 2.5] \)
  since we anticipate the explicit scheme to limit stability at \( C_{s,x} \approx 1, \Rightarrow \) time-step for resolving waves with \( c_s \) over given \( \Delta x \); and
- \( \frac{C_{s,z}}{\Delta t^*} = \frac{k_z}{k_x} \in [10^{-2}, 10^4] \),
  since we need stability to be ensured using \( \Delta t^* \) for the largest vertical Courant numbers, which depend on model resolution.
Linear analyses

We numerically generate values for $\mathbf{A}$ (and $\Rightarrow (\lambda_1, \lambda_2, \lambda_3)$) and consider:

- amplitude errors: $\Rightarrow$ instability?
- phase errors: $\Rightarrow$ implied direction of group velocity?

We consider the acoustic Courant number ($C_{s,x} \equiv c_s k_x \Delta t$, $C_{s,z} \equiv c_s k_z \Delta t$) ranges:

- $C_{s,x} \equiv \Delta t^* \in [0, 2.5]$ since we anticipate the explicit scheme to limit stability at $C_{s,x} \approx 1$, $\Rightarrow$ time-step for resolving waves with $c_s$ over given $\Delta x$; and

- $\frac{C_{s,z}}{\Delta t^*} = \frac{k_z}{k_x} \in [10^{-2}, 10^4]$, since we need stability to be ensured using $\Delta t^*$ for the largest vertical Courant numbers, which depend on model resolution. Typically:

\[
\Delta x \approx \begin{cases} 
10^3 \text{ m, for high -- res weather} \\
10^5 \text{ m, for climate}
\end{cases}
\quad \text{and} \quad \Delta z = \begin{cases} 
\Delta z_B \approx 10 \text{ m, model bottom} \\
\Delta z_T \approx 10^3 \text{ m, model top}
\end{cases}
\]

$\Rightarrow$ to resolve the largest $C_{s,z}$ with $\Delta t^*$, we must consider

\[
\frac{\max C_{s,z}}{\Delta t^*} = \frac{k_{z\text{max}}}{k_{x\text{max}}} = \frac{\Delta x}{\Delta z_{\text{min}}} \leq \begin{cases} 
10^2 \text{ for high -- res weather} \\
10^4 \text{ for climate}
\end{cases}
\]
Example amplitude and phase errors: UJ3(1,3,2)

From Ullrich & Jablonowski (2012): “Strang carryover”:

<table>
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<tr>
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<td>0</td>
</tr>
</tbody>
</table>

- SPP 3rd order;
- 1 implicit stage;
- 3 explicit stages;
- overall, 2nd order;
- time-levels: completely split;
- final stage: implicit.
Example amplitude and phase errors: UJ3(1,3,2)

From Ullrich & Jablonowski (2012): “Strang carryover”:

|   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 1/2 | 1/4 | 1/4 | 0   | 0   | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0 | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 1/2 | 1/4 | 1/4 | 0   | 0   | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 |
| 1 | 0   | 0   | 1/6 | 1/6 | 2/3 | 0   | 1/6 | 1/6 | 1/6 | 1/6 | 2/3 | 0   | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |
| 1/2| 1/6 | 1/6 | 2/3 | 0   | 1   | 0   | 1/6 | 1/6 | 1/6 | 1/6 | 2/3 | 0   | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |

Amplitudes of the two acoustic modes:

Amplitudes of the non-divergent mode remain neutral in the stable region.
Example amplitude and phase errors: UJ3(1,3,2)

From Ullrich & Jablonowski (2012): “Strang carryover”:

\[
\begin{array}{c|cccc}
0 & 0 & 0 & 1 & 1/2 \\
0 & 0 & 0 & 1/6 & 1/6 \\
1/2 & 0 & 1/4 & 1/6 & 2/3 \\
1 & 0 & 1/6 & 1/6 & 2/3 \\
1 & 0 & 1/6 & 1/6 & 2/3 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
0 & 0 & 0 & 1/2 & 1/2 \\
1/2 & 1/2 & 0 & 1/2 & 1/2 \\
1 & 2/3 & 0 & 1/2 & 1/2 \\
1 & 2/3 & 0 & 1/2 & 1/2 \\
1 & 2/3 & 0 & 1/2 & 1/2 \\
1 & 2/3 & 0 & 1/2 & 1/2 \\
\end{array}
\]

Phase of the two acoustic modes:

**True phase:**
- two acoustic modes: constant phase/group velocity wrt $k_x$ or $k_z$;
- non-divergent mode: zero phase everywhere.
Example amplitude and phase errors: UJ3(1,3,2)

From Ullrich & Jablonowski (2012): “Strang carryover”:

Phase of the two acoustic modes:

Phase of the non-divergent mode remains zero in the stable region.
Linear analyses: $|A|$ for acoustic modes (UFPreF)

Good stability properties; some very strong damping; some asymmetry in acoustic modes
Linear analyses: $|A|$ for acoustic modes (UFPreB)

Generally, UFPreB ⇒ greater stability than UFPreF, except “ARS” schemes
Linear analyses: $\theta$ for acoustic modes (UFPreB)

Good representation of phase for small Courant numbers; some evidence of group velocity reversal, but only close to limits of stability.
Numerical results (Weller et al., 2013)

- Non-linear near-hydrostatic Boussinesq test (Durran & Blossey, 2012)
- Localized forcing ($\nabla \times \psi$) generating gravity waves in stratified shear flow

\[ u_0(z) = 5 + z + 0.4(5 - z)(5 + z) \text{ m s}^{-1} \]

\[ \psi(x, z, t) = \psi_0 \left( \frac{\pi x}{L_x} \right) \sin(\omega t) \exp \left[ - \left( \frac{\pi x}{L_x} \right)^2 - \left( \frac{\pi z}{L_z} \right)^2 \right] \text{ m}^2 \text{s}^{-1} \]

\[ \omega = 1.25 \times 10^{-4} \text{ s}^{-1}, \quad L_x = 160 \text{ km}, \quad L_z = 10 \text{ km}, \quad \psi_0 = 10 \text{ m}^2 \text{s}^{-1} \]

\[ \Delta x = 10 \text{ km}, \quad \Delta z = 250 \text{ m} \]

![Diagram of buoyancy, velocity vectors, and forcing streamfunction](image)
Numerical results: RMS buoyancy error (33.3h)
## Indication of relative computational cost

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Total $(v)$</th>
<th>Implicit $(s)$</th>
<th>$\Delta t^*_{\text{max}}$</th>
<th>$v/\Delta t^*_{\text{max}}$</th>
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- $v$ and $s$: number of RHS computations (total and implicit respectively) required per $\Delta t$;
- $\Delta t^*_{\text{max}}$: largest $\Delta t^*$ for which $|A| \leq 1$ for all $k_z/k_x$;

$\Rightarrow v/\Delta t^*_{\text{max}}$ indicates relative cost
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Horizontally-explicit vertically-implicit (HEVI) time-stepping methods for NWP and climate models

Sarah-Jane Lock

Thanks for your attention!