Scale-dependent time integration and thermodynamic consistency for weakly compressible flows

... or ...

Rupert Klein

Mathematik & Informatik, Freie Universität Berlin
Towards a
“very balanced” compressible flow solver

Rupert Klein

Mathematik & Informatik, Freie Universität Berlin
Thanks to...

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Limit regimes in atmospheric flows

Sound-proof limits

Semi-implicit scheme for compressible flows

Scale-dependent time integration

Extensions: Moisture & general Eqs. of State
Asymptotic Modelling Framework

10 km / 20 min

1000 km / 2 days

10000 km / 1 season

Thanks to:
P.K. Taylor, Southampton Oceanogr. Inst.; P. Névir, Freie Universität Berlin; S. Rahmstorf, PIK, Potsdam
Asymptotic Modelling Framework

Anelastic Boussinesque Model

\[ \begin{align*}
    &u_t + u \cdot \nabla u + w u_z + \nabla \pi = S_u \\
    &w_t + u \cdot \nabla w + w w_z + \pi_z = -\theta' + S_w \\
    &\theta'_t + u \cdot \nabla \theta' + w \theta'_z = S'_\theta \\
    &\nabla \cdot (\rho_0 u) + (\rho_0 w)_z = 0 \\
    &\theta = 1 + \varepsilon^4 \theta'(x, z, t) + o(\varepsilon^4)
\end{align*} \]

10 km / 20 min

Quasi-geostrophic theory

\[ \begin{align*}
    &q = \zeta^{(0)} + \Omega_0 \beta \eta + \Omega_0 \frac{\partial}{\partial z} \left( \frac{\rho^{(0)}}{\rho_0} \frac{d\Theta}{dz} \theta^{(3)} \right) \\
    &\zeta^{(0)} = \nabla^2 \pi^{(3)}, \quad \theta^{(3)} = -\frac{\partial \pi^{(3)}}{\partial z}, \quad u^{(0)} = \frac{1}{\Omega_0} k \times \nabla \pi^{(3)}
\end{align*} \]

1000 km / 2 days

EMIC - equations (CLIMBER-2)

\[ \begin{align*}
    &\frac{\partial Q_T}{\partial t} + \nabla \cdot F_T = S_T \\
    &\frac{\partial Q_q}{\partial t} + \nabla \cdot F_q = S_q \\
    &Q_T = \int_{\rho_0}^{\rho} \int_{\rho_0}^{\rho} \left( \rho u \left( \varphi + u \varphi \right) + \varphi \right) \, d\rho \\
    &F_T = \int_{\rho_0}^{\rho} \int_{\rho_0}^{\rho} \left( \rho u \left( \varphi + u \varphi \right) + \varphi \right) \, d\rho \\
    &Q_q = \int_{\rho_0}^{\rho} \int_{\rho_0}^{\rho} \left( \rho u \left( \varphi + u \varphi \right) + \varphi \right) \, d\rho \\
    &F_q = \int_{\rho_0}^{\rho} \int_{\rho_0}^{\rho} \left( \rho u \left( \varphi + u \varphi \right) + \varphi \right) \, d\rho \\
    &u = u_0 + u_\alpha, \quad f_\rho \times u_0 = -\nabla \pi, \quad u_\alpha = \alpha \nabla p_0
\end{align*} \]

V. Petoukhov et al., CLIMBER-2 ..., Climate Dynamics, 16, (2000)

10000 km / 1 season
# Asymptotic Modelling Framework

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth’s radius</td>
<td>$a \sim 6 \cdot 10^6 \text{ m}$</td>
</tr>
<tr>
<td>Earth’s rotation rate</td>
<td>$\Omega \sim 10^{-4} \text{ s}^{-1}$</td>
</tr>
<tr>
<td>Acceleration of gravity</td>
<td>$g \sim 9.81 \text{ ms}^{-2}$</td>
</tr>
<tr>
<td>Sea level pressure</td>
<td>$p_{\text{ref}} \sim 10^5 \text{ kgm}^{-1}\text{s}^{-2}$</td>
</tr>
<tr>
<td>H$_2$O freezing temperature</td>
<td>$T_{\text{ref}} \sim 273 \text{ K}$</td>
</tr>
<tr>
<td>Tropospheric potential temperature variation</td>
<td>$\Delta \Theta \sim 40 \text{ K}$</td>
</tr>
<tr>
<td>Dry gas constant</td>
<td>$R \sim 287 \text{ m}^2\text{s}^{-2}\text{K}^{-1}$</td>
</tr>
<tr>
<td>Dry isentropic exponent</td>
<td>$\gamma \sim 1.4$</td>
</tr>
</tbody>
</table>

## Distinguished limit:  

$$\Pi_1 = \frac{h_{\text{sc}}}{a} \sim 1.6 \cdot 10^{-3} \sim \varepsilon^3$$  

$$\Pi_2 = \frac{\Delta \Theta}{T_{\text{ref}}} \sim 1.5 \cdot 10^{-1} \sim \varepsilon$$  

$$\Pi_3 = \frac{c_{\text{ref}}}{\Omega a} \sim 4.7 \cdot 10^{-1} \sim \sqrt{\varepsilon}$$

### Where:

$$h_{\text{sc}} = \frac{RT_{\text{ref}}}{g} = \frac{p_{\text{ref}}}{\rho_{\text{ref}}g} \sim 8.5 \text{ km}$$  

$$c_{\text{ref}} = \sqrt{RT_{\text{ref}}} = \sqrt{gh_{\text{sc}}} \sim 300 \text{ m/s}$$
Asymptotic Modelling Framework


distinguished limit continued

\[ \text{Fr}_{\text{int}} \sim \epsilon \]
\[ \text{Ro}_{h_{sc}} \sim \epsilon^{-1} \]
\[ \text{Ro}_{L_{Ro}} \sim \epsilon \]
\[ \text{Ma} \sim \epsilon^{3/2} \]
Asymptotic Modelling Framework

Compressible flow equations with general source terms

\[
\left( \frac{\partial}{\partial t} + v_{\|} \cdot \nabla_{\|} + w \frac{\partial}{\partial z} \right) v_{\|} + \varepsilon (2\Omega \times v)_{\|} + \frac{1}{\varepsilon^3 \rho} \nabla_{\|} p = S_{v_{\|}},
\]

\[
\left( \frac{\partial}{\partial t} + v_{\|} \cdot \nabla_{\|} + w \frac{\partial}{\partial z} \right) w + \varepsilon (2\Omega \times v)_{\perp} + \frac{1}{\varepsilon^3 \rho} \frac{\partial p}{\partial z} = S_w - \frac{1}{\varepsilon^3},
\]

\[
\left( \frac{\partial}{\partial t} + v_{\|} \cdot \nabla_{\|} + w \frac{\partial}{\partial z} \right) \rho + \rho \nabla \cdot v = 0,
\]

\[
\left( \frac{\partial}{\partial t} + v_{\|} \cdot \nabla_{\|} + w \frac{\partial}{\partial z} \right) \Theta = S_\Theta.
\]

Expansions

\[
\begin{pmatrix}
\rho \\
v_{\|} \\
\rho \\
\Theta
\end{pmatrix} =: \mathbf{U} = \sum_{i=0}^{m} (\varepsilon^\alpha)^i \mathbf{U}^{(i)} + o \left( (\varepsilon^\alpha)^m \right)
\]
Recovered classical single-scale models:

\[ U^{(i)} = U^{(i)}(t, x, z) \]  
Linear small scale internal gravity waves

Anelastic & pseudo-incompressible models

\[ U^{(i)} = U^{(i)}(t, x, z) \]

Linear large scale internal gravity waves

Mid-latitude Quasi-Geostrophic Flow

\[ U^{(i)} = U^{(i)}(\epsilon^2 t, \epsilon^2 x, z) \]

Equatorial Weak Temperature Gradients

\[ U^{(i)} = U^{(i)}(\epsilon^2 t, \epsilon^{-1} \xi(\epsilon^2 x), z) \]  
Semi-geostrophic flow

Kelvin, Yanai, Rossby, and gravity Waves

\[ U^{(i)} = U^{(i)}(\epsilon^{5/2} t, \epsilon^{5/2} x, \epsilon^{5/2} y, z) \]

... and many more
Asymptotic Modelling Framework

\[ \frac{h_{sc}}{u_{ref}} \]

\[ 1/\varepsilon^3 \]
\[ 1/\varepsilon^{5/2} \]
\[ 1/\varepsilon^2 \]
\[ 1/\varepsilon \]
\[ \varepsilon \]

bulk micro convective meso synoptic planetary

\[ 1 \]

\[ \varepsilon \]

advection acoustic waves inertial waves

\[ 1/\varepsilon \]

\[ 1/\varepsilon^2 \]

\[ 1/\varepsilon^3 \]

\[ h_{sc} \]

Boussinesq WTG HPE

anelastic / pseudo-incompressible internal waves

WTG + Coriolis HPE + Coriolis

PG

Obukhov scale

\[ 1/\varepsilon^{5/2} \]

Limit regimes in atmospheric flows

**Sound-proof limits**

Semi-implicit scheme for compressible flows

Scale-dependent time integration

Extensions: Moisture & general Eqs. of State
Key question:

What is the slow flow limiting dynamics like?

i.e.

What should a compressible solver do in the limit?
Sound-Proof Models

Compressible & sound-proof flow equations

\[ \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \]

\[ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{u}) + P \nabla \| \pi = 0 \]

\[ (\rho \mathbf{w})_t + \nabla \cdot (\rho \mathbf{v} \mathbf{w}) + P \pi_z = -\rho g \]

\[ P_t + \nabla \cdot (P \mathbf{v}) = 0 \]

\[ P = \rho^{1 \over \gamma} = \rho \theta, \quad \pi = p / \Gamma P, \quad \Gamma = c_p / R, \quad \mathbf{v} = \mathbf{u} + \mathbf{w} \mathbf{k} \quad (\mathbf{u} \cdot \mathbf{k} \equiv 0) \]

Drop term for:

anelastic\(^\dagger\) (approx.)

pseudo-incompressible\(^*\)

(hydrostatic-primitive)

Parameter range & length and time scales of asymptotic validity?

\(^\dagger\) e.g. Lipps & Hemler, JAS, 29, 2192–2210 (1982)

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\[ (\rho \mathbf{w})_t + \nabla \cdot (\rho \mathbf{v} \cdot \mathbf{w}) + P\pi_z = -\rho g \]

\[ P_t + \nabla \cdot (P \mathbf{v}) = 0 \]

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drop term for:
anelastic\(^\dagger\) (approx.)
pseudo-incompressible\(^*\)
(hydrostatic-primitive)

e.g. Lipps & Hemler, JAS, 29, 2192–2210 (1982)
From here on $\varepsilon$ is the (isothermal) Mach number

$$\varepsilon = \frac{u_{\text{ref}}}{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}} = \frac{u_{\text{ref}}}{\sqrt{gh_{\text{sc}}}}$$
Characteristic (inverse) time scales

**advection** : \( \frac{u_{\text{ref}}}{h_{\text{sc}}} \)  \( \frac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}} = 1 \)

**internal waves** : \( N = \sqrt{\frac{g}{\theta}} \frac{d\theta}{d\bar{z}} \)  \( \frac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}} \frac{h_{\text{sc}} d\theta}{\theta d\bar{z}} = \frac{1}{\varepsilon} \frac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}} \frac{h_{\text{sc}} d\theta}{\theta d\bar{z}} \)

**sound** : \( \frac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \frac{\sqrt{gh_{\text{sc}}}}{h_{\text{sc}}} \)  \( \frac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}} = 1 \frac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}} \)
Design Regime (10 km / 20 min)

Characteristic (inverse) time scales

<table>
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<tr>
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Ogura & Phillips’ regime* with two time scales

\( \bar{\theta} = 1 + \varepsilon^2 \bar{\theta}(z) + \ldots \) \( \Rightarrow \) \( \frac{h_{\text{sc}}\,d\bar{\theta}}{\bar{\theta}\,dz} = O(\varepsilon^2) \)

Design Regime (10 km / 20 min)

Characteristic (inverse) time scales

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| internal waves | \( N = \sqrt{\frac{g d\hat{\theta}}{\theta} dz} \) | \( \sqrt{gh_{sc}} \) \( \frac{h_{sc} d\hat{\theta}}{\theta} dz \) = \( \sqrt{h_{sc} d\hat{\theta}} \)
| sound       | \( \frac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{sc}} = \sqrt{gh_{sc}} \) | \( \frac{\sqrt{gh_{sc}}}{u_{\text{ref}}} \) = \( \frac{1}{\varepsilon} \) |

Ogura & Phillips’ regime* with two time scales

\[ \bar{\theta} = 1 + \varepsilon^2 \hat{\theta}(z) + \ldots \quad \Rightarrow \quad \frac{h_{sc} d\bar{\theta}}{\theta} dz = O(\varepsilon^2) \quad \Rightarrow \quad \Delta \bar{\theta} \bigg|_{z=0} < 1 \text{ K} \]

Design Regime (10 km / 20 min)

### Characteristic (inverse) time scales

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\( \sqrt{gh_{sc}} \sqrt{\frac{d\theta}{\theta dz}} \) = \frac{1}{\epsilon^{\nu}} \sqrt{\frac{h_{sc} d\theta}{\theta dz}}

Realistic regime with three time scales

\( \bar{\theta} = 1 + \epsilon^{\mu} \hat{\theta}(z) + \ldots \) \quad \Rightarrow \quad \frac{h_{sc} d\theta}{\theta dz} = O(\epsilon^{\mu}) \quad (\nu = 1 - \mu/2)
Fast linear compressible / pseudo-incompressible modes

\[ \tilde{\theta}_\vartheta + \tilde{w} \frac{d\tilde{\theta}}{dz} = 0 \]

\[ \tilde{\vartheta}_\vartheta + \tilde{\theta} k + \tilde{\theta} \nabla \pi^* \] \[ + \varepsilon^\mu \pi^*_\vartheta + \left( \gamma \Gamma \pi \nabla \cdot \tilde{\vartheta} + \tilde{w} \frac{d\pi}{dz} \right) = 0 \]

Vertical mode expansion (separation of variables)

\[ \left( \begin{array}{c} \tilde{\theta} \\ \tilde{u} \\ \tilde{w} \\ \pi^* \end{array} \right) (\vartheta, \varphi, z) = \left( \begin{array}{c} \Theta^* \\ U^* \\ W^* \\ \Pi^* \end{array} \right) (z) \exp \left( i [\omega \vartheta - \lambda \cdot \varphi] \right) \]
Design Regime (10 km / 20 min)

\[- \frac{d}{dz} \left( \frac{1}{1 - \frac{\varepsilon \mu \omega^2}{\lambda^2 c^2}} \frac{dW^*}{dz} \right) + \frac{\lambda^2}{\theta P} W^* = \frac{1}{\omega^2} \frac{\lambda^2 N^2}{\theta P} W^* \]

Internal wave modes \( \left( \frac{\omega^2}{c^2} = O(1) \right) \)

- pseudo-incompressible modes/EVals = compressible modes/EVals + \( O(\varepsilon^\mu) \)

- phase errors remain small over advection time scales for \( \mu > \frac{2}{3} \)

The anelastic and pseudo-incompressible models remain relevant for stratifications

\[ \frac{1}{\theta} \frac{d\bar{\theta}}{dz} < O(\varepsilon^{2/3}) \quad \Rightarrow \quad \Delta \theta \bigg|_{h_{sc}} \lesssim 40 \text{ K} \]

not merely up to \( O(\varepsilon^2) \) as in Ogura-Phillips (1962)
Key question:

What is the slow flow limiting dynamics like?

i.e.

What should a compressible solver do in the limit?

Answer:

Behave pseudo-incompressibly!*

* Anelastic “looses” only for breaking of internal wave packets in the stratosphere
Limit regimes in atmospheric flows

Sound-proof limits

**Semi-implicit scheme for compressible flows**

Scale-dependent time integration

Extensions: Moisture & general Eqs. of State
pseudo-incompressible ⇔ compressible
Pseudo-incompressible $\Leftrightarrow$ compressible

Compressible

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) + P \nabla \pi &= -\rho g \mathbf{k} \\
\mathbf{P}_t + \nabla \cdot (P \mathbf{v}) &= 0
\end{align*}
\]

\[
P = p^{\frac{1}{\gamma}} = \rho \theta , \quad \pi = p / \Gamma P , \quad \Gamma = c_p / R , \quad \mathbf{v} = \mathbf{u} + w \mathbf{k} \quad (\mathbf{u} \cdot \mathbf{k} \equiv 0)
\]
Pseudo-incompressible $\iff$ compressible

Pseudo-incompressible

\[
\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0
\]

\[
(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) + \overline{P} \nabla \pi = -\rho g \mathbf{k}
\]

\[
\times \quad \nabla \cdot (\overline{P} \mathbf{v}) = 0
\]

\[
\rho \theta = \overline{P}, \quad \pi : \text{“elliptic pressure”}
\]
 Predictor-corrector scheme* for pseudo-incompressible flow
 Predictor

Solve auxiliary hyperbolic system over \( t^n \rightarrow t^{n+1} \)
(by your favorit 2nd order scheme)*

\[
\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0
\]

\[
(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) = -\rho g k - P \nabla \pi^n
\]

\[
P_t + \nabla \cdot (P \mathbf{v}) = 0
\]

Predicted values satisfy

\[
\begin{pmatrix}
\rho \\
P \\
\theta
\end{pmatrix}^{n+1,*} = \begin{pmatrix}
\rho \\
P \\
\theta
\end{pmatrix}^{n+1} + O \left((\Delta t)^3\right)
\]
**Predictor**

Solve auxiliary hyperbolic system over \( t^n \to t^{n+1} \)

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\[
\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0
\]

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(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) = -\rho g k - P \nabla \pi^n
\]

\[
P_t + \nabla \cdot (P \mathbf{v}) = 0
\]

But

\[
\mathbf{v}^{n+1,*} = \mathbf{v}^{n+1} + O \left( (\Delta t)^2 \right)
\]

\[
P^{n+1,*} \neq \overline{P}
\]
Corrector for advective fluxes

\[ \pi^{n+1} = \pi^n + \delta\pi \]

\[ (P\nu)^{n+1/2} = (P\nu)^{n+1/2,*} - \frac{\Delta t}{2} P\theta \nabla \delta\pi \]

\[ P^{n+1} = P^n - \Delta t \nabla \cdot (P\nu)^{n+1/2} \equiv \overline{P} \]
Corrector for advective fluxes

\[ \pi^{n+1} = \pi^n + \delta \pi \]

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\[ P^{n+1} = P^{n+1,*} + \frac{(\Delta t)^2}{2} \nabla \cdot (P\theta \nabla \delta \pi) \]

Solve elliptic pressure equation

\[ \nabla \cdot (P\theta \nabla \delta \pi) = \frac{2}{(\Delta t)^2} \left( \bar{P} - P^{n+1,*} \right) \]
Corrector for advective fluxes

\[ \pi^{n+1} = \pi^n + \delta\pi \]

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\[ P^{n+1} = P^{n+1,*} + \frac{(\Delta t)^2}{2} \nabla \cdot (P\theta \nabla \delta\pi) = \bar{P} \]

Flux correction for advected scalars \( X \in \{1, 1/\theta, \nu/\theta\} \)

\[ (PX)^{n+1} = (PX)^{n+1,*} + \frac{(\Delta t)^2}{2} \nabla \cdot (X P\theta \nabla \delta\pi) \]
That’s it up to ...

divergence control for $\nu^{n+1}$

some “bells & whistles”
Predictor-corrector scheme

for

compressible flow
Predictor*

Solve auxiliary hyperbolic system over $t^n \rightarrow t^{n+1}$
(by your favorit 2nd order scheme)

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) = -\rho g \mathbf{k} - P \nabla \pi^n$$

$$P_t + \nabla \cdot (P \mathbf{v}) = 0$$

Predicted values satisfy

$$\begin{pmatrix} \rho \\ P \\ \theta \end{pmatrix}^{n+1,*} = \begin{pmatrix} \rho \\ P \\ \theta \end{pmatrix}^{n+1} + O \left( (\Delta t)^3 \right)$$

*Identical to psinc-predictor!
Predictor*

Solve auxiliary hyperbolic system over \( t^n \rightarrow t^{n+1} \)
(by your favorit 2nd order scheme)

\[
\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0
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\]

\[
P_t + \nabla \cdot (P \mathbf{v}) = 0
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But

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\mathbf{v}^{n+1,*} = \mathbf{v}^{n+1} + O\left((\Delta t)^2\right)
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\[ (PX)^{n+1} = (PX)^{n+1,*} + \frac{(\Delta t)^2}{2} \nabla \cdot (X P\theta \nabla \delta \pi) \]

*Identical to psinc-predictor!
Corrector for advective fluxes

$$\pi^{n+1} = \pi^n + \delta\pi$$

$$(P\mathbf{v})^{n+1/2} = (P\mathbf{v})^{n+1/2,*} - \frac{\Delta t}{2} P\theta \nabla \delta\pi$$

$$P^{n+1} = P^n - \Delta t \nabla \cdot (P\mathbf{v})^{n+1/2}$$

But now

$$P^{n+1} - P^n = \left(\frac{\partial P}{\partial \pi}\right)^{n+1/2} \delta\pi + O\left((\delta\pi)^3\right)$$
Corrector for advective fluxes

\[ \pi^{n+1} = \pi^n + \delta\pi \]

\[
(P\nu)^{n+1/2} = (P\nu)^{n+1/2,*} - \frac{\Delta t}{2} P\theta \nabla \delta\pi
\]

\[ P^{n+1} = P^n - \Delta t \nabla \cdot (P\nu)^{n+1/2} \]

Solve Helmholtz equation

\[
\frac{2}{(\Delta t)^2} \left( \frac{\partial P}{\partial \pi} \right)^{n+1/2} \delta\pi - \nabla \cdot (P\theta \nabla \delta\pi) = \frac{2}{(\Delta t)^2} \left( P^{n+1,*} - P^n \right)
\]
Corrector for advective fluxes

\[ \pi^{n+1} = \pi^n + \delta \pi \]

\[ (P\nu)^{n+1/2} = (P\nu)^{n+1/2,*} - \frac{\Delta t}{2} P\theta \nabla \delta \pi \]

\[ P^{n+1} = P^n - \Delta t \nabla \cdot (P\nu)^{n+1/2} \]

Exner pressure post-correction

\[ \pi^{n+1} = \frac{1}{\Gamma} \left( P^{n+1} \right)^{\gamma-1} \]
Bells & Whistles

• Well-balanced discretization of gravity term / no background state
  ([1] Botta et al., JCP, 196, 539-565, (2004))

• Positivity of advection in spatial op-split mode

• Runge-Kutta, MUSCL-type, BDF2 predictor time integrators available

• Inf-Sup-stable version of projection step
Some results
Diagonally advected vortex

\begin{align*}
t &= 0 \\
& \quad \text{density} \quad \text{vorticity} \\
\quad \text{density} \quad \text{vorticity} \\
\quad \text{density} \quad \text{vorticity} \\
\end{align*}
Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997)) ⇒ Joana’s talk!

Absorption layers
\[ d = 0 \ldots 1/600s \]

\[
\begin{align*}
U &= 10 \text{ m/s} \\
N &= 0.01 \text{ 1/s}
\end{align*}
\]

“Witch of Agnesi”
\[ h = 628 \text{ m}, \ l = 1000\text{m} \]
Results at time $t = 2h$

pseudo-incompressible

compressible, $CFL_{\text{adv}} = 1$

compressible, $CFL_{\text{ac}} = 2$
Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))

Compressible Euler eqs.

3 hours

sharpened van Leer’s limiter

$\Delta t \cdot \text{residual} < 10^{-4}$

CFL$_{adv} = 1.0$
Thermodynamically consistent "psinc"

Standard model in conservative form

\[ \rho^* t + \nabla \cdot (\rho^* \mathbf{v}) = 0 \]

\[ (\rho^* \mathbf{u}) t + \nabla \cdot (\rho^* \mathbf{v} \circ \mathbf{u}) + P \nabla || \pi = 0 \]

\[ (\rho^* \mathbf{w}) t + \nabla \cdot (\rho^* \mathbf{v} \mathbf{w}) + \rho^* \theta \pi z = -\rho^* g \]

\[ \nabla \cdot (\bar{P} \mathbf{v}) = 0 \]

\[ \rho^* \theta = \bar{P}, \quad \pi = \bar{\pi}(z) + \pi', \quad \mathbf{v} = \mathbf{u} + \omega \mathbf{k} \quad (\mathbf{u} \cdot \mathbf{k} \equiv 0) \]

\[ \rho^* \text{ is Durran's "pseudo-density"} \]
Thermodynamically consistent "psinc"

Standard model in momentum-advective form*

\[ \rho^*_t + \nabla \cdot (\rho^* \mathbf{v}) = 0 \]

\[ \mathbf{u}_t + \mathbf{v} \cdot \nabla \mathbf{u} + \theta \nabla || \pi' = 0 \]

\[ w_t + \mathbf{v} \cdot \nabla w + \theta \pi'_z = g \frac{\theta - \bar{\theta}}{\bar{\theta}} = g \frac{\theta'}{\bar{\theta}} \]

\[ \nabla \cdot (P \mathbf{v}) = 0 \]

\( \rho^* \) is the density effective in the momentum equation!

Thermodynamically consistent "psinc"

Thermodynamically consistent* model in conservative form

\[ \rho^*_t + \nabla \cdot (\rho^* v) = 0 \]

\[ (\rho^* u)_t + \nabla \cdot (\rho^* v \circ u) + \nabla \parallel p = 0 \]

\[ (\rho^* w)_t + \nabla \cdot (\rho^* vw) + p_z = - \left( \rho^* + \frac{\partial \rho}{\partial p} p' \right) g \]

\[ \nabla \cdot (\bar{P} v) = 0 \]

\[ \rho^* \theta = \bar{P} , \quad p = \bar{p}(z) + p' , \quad v = u + w k \quad (u \cdot k \equiv 0) \]

Straka’s test

compressible (COMP)

\( p, p \)-formulation (PI)

thermodynamically consistent (TC)*
Straka’s test – model comparison
Limit regimes in atmospheric flows

Sound-proof limits

Semi-implicit scheme for compressible flows

**Scale-dependent time integration**

Extensions: Moisture & general Eqs. of State
Scale-dependent time integration

Why not simply solve the full compressible equations?

Competing approaches:

- Split-explicit / multi-rate methods, e.g.,
  - Runge-Kutta (slow) + forward-backward (fast), e.g.,
    Wicker & Skamarock, MWR, (98), ... ;
    MM5, LM, WRF ... 
  - Multirate infinitesimal schemes, peer methods
    Wensch et al., BIT, (09);
    ASAM, ...

- Semi-implicit / linearly implicit schemes
  - explicit advection, damped 2nd or 1st-order schemes for fast modes, e.g.,
    Robert, Japan Met. J., (69), ... ;
    UKMO, ...
  - linearly implicit Rosenbrock-type methods, e.g.,
    Reisner et al., MWR, (05), ...;
    ASAM, LANL Hurricane model, ...

- Fully implicit integration
Scale-dependent time integration

Why not simply solve the full compressible equations?

Linear acoustics, simple wave initial data, periodic domain

*(integration: implicit midpoint rule, staggered grid, 512 grid pts., CFL = 10)*

![Graphs showing pressure changes over time](image)

- $t = 0$
- $t = 3$
Scale-dependent time integration

Why not simply solve the full compressible equations?

Linear acoustics, simple wave initial data, periodic domain
(integration: implicit midpoint rule, staggered grid, 512 grid pts., CFL = 10)

Ideas:

- Slave short waves \((c\Delta t/\ell > 1)\) to long waves \((c\Delta t/\ell \leq 1)\)
- with pseudo-incompressible limit behavior

“super-implicit” scheme
non-standard multi grid
projection method
Scale-dependent time integration

\[ \varepsilon \ddot{y} + \varepsilon \kappa \dot{y} + y = \cos(t), \quad \begin{cases} y(0) = 1 + a \\ \dot{y}(0) = 0 \end{cases}, \quad (\varepsilon = 0.01) \]
Scale-dependent time integration

\[ \varepsilon \ddot{y} + \varepsilon \kappa \dot{y} + y = \cos(t) \]

**Slow-time asymptotics for \( \varepsilon \ll 1 \):**

\[ y(t) = y^{(0)}(t) + \varepsilon y^{(1)}(t) + \ldots, \quad y^{(0)}(t) = \cos(t) \]
\[ y^{(1)}(t) = -(\ddot{y}^{(0)} + \kappa \dot{y}^{(0)})(t) \]

**Associated “super-implicit” discretization (extreme BDF):**

\[ y^{n+1} = \cos(t^{n+1}) - \varepsilon [ (\delta_t + \kappa) \dot{y} ]^{*,n+1} \]
\[ \dot{y}^{n+1} = \frac{1}{\Delta t} \left( y^{n+1} - y^n + \frac{1}{2} (y^{n+1} - 2y^n + y^{n-1}) \right) \]

where

\[ u^{*,n+1} = 2u^n - u^{n-1} \]
\[ (\delta_t u)^{*,n+1} = \frac{1}{\Delta t} \left( u^n - u^{n-1} + \frac{3}{2} (u^n - 2u^{n-1} + u^{n-2}) \right) \]
Scale-dependent time integration

Implicit midpoint rule \[ \Delta t = 7 \sqrt{\varepsilon} \]

Super-implicit scheme

\[ \Delta t = 7 \sqrt{\varepsilon} \]
Scale-dependent time integration

Implicit midpoint rule \( \Delta t = 5.55\sqrt{\varepsilon} \)

\( \Delta t = 5.55\sqrt{\varepsilon} \)  Super-implicit scheme
Scale-dependent time integration

**Blended scheme** \( \Delta t = 5.55 \sqrt{\varepsilon} \)

\[
\Delta y\big|_{BL} = \eta \Delta y\big|_{IMP} + (1 - \eta) \Delta y\big|_{SupI}
\]

\( \Delta t = 5.55 \sqrt{\varepsilon} \) BDF2 – for comparison
Compressible flow equations:

\[ \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \]

\[ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) + P \nabla \pi = -\rho g \mathbf{k} \]

\[ P_t + \nabla \cdot (P \mathbf{v}) = 0 \]

\[ P = p^{\frac{1}{\gamma}} = \rho \theta, \quad \pi = p/\Gamma P, \quad \Gamma = c_p/R \]
Scale-dependent time integration

For starters: **1D Linear acoustics:**

\[ u_t + p_x = 0 \]
\[ p_t + c^2 u_x = 0 \]

Desired:

- remove underresolved modes
- minimize dispersion for marginally resolved modes
Scale-dependent time integration

1D Linear acoustics:

\[
\begin{align*}
  u_t + p_x &= 0 \\
p_t + c^2 u_x &= 0
\end{align*}
\]

Desired:
- remove underresolved modes
- minimize dispersion for marginally resolved modes

Strategy:

scale-dependent IMP-SupI-Blended scheme via multi grid
Scale-dependent time integration

**Implicit mid-point rule for linear acoustics**

\[
\frac{u^{n+1} - u^n}{\Delta t} + \frac{\partial}{\partial x} p^{n+\frac{1}{2}} = 0, \quad \frac{p^{n+1} - p^n}{\Delta t} + c^2 \frac{\partial}{\partial x} u^{n+\frac{1}{2}} = 0
\]

with

\[
X^{n+\frac{1}{2}} = \frac{1}{2} \left( X^{n+1} + X^n \right)
\]

**Implicit problem for half-time fluxes**

\[
u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2} \frac{\partial}{\partial x} p^{n+\frac{1}{2}}, \quad p^{n+\frac{1}{2}} = p^n - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^{n+\frac{1}{2}}
\]

Eliminate \( u^{n+\frac{1}{2}} \)

\[
\left( 1 - \frac{c^2 \Delta t^2}{4} \frac{\partial^2}{\partial x^2} \right) p^{n+\frac{1}{2}} = p^n - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^n
\]
Scale-dependent time integration

**Implicit mid-point rule** \( \Rightarrow \) **super-implicit**

\[
\begin{align*}
    u^{n+\frac{1}{2}} &= u^n - \frac{\Delta t}{2} \frac{\partial}{\partial x} p^{n+\frac{1}{2}} \\
    p^{n+\frac{1}{2}} &= p^n - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^{n+\frac{1}{2}}
\end{align*}
\]

key step:

\[
\begin{align*}
    u^{n+\frac{1}{2}} &= u^n - \frac{\Delta t}{2} \frac{\partial}{\partial x} p^{n+\frac{1}{2}} \\
    &= - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^{n+\frac{1}{2}} - \frac{\Delta t}{2} \left( \frac{\partial p}{\partial t} \right)^{BD,n+\frac{1}{2}}
\end{align*}
\]

Pressure “projection” equation

\[
\frac{c^2 \Delta t}{2} \frac{\partial^2}{\partial x^2} p^{n+\frac{1}{2}} = c^2 \frac{\partial}{\partial x} u^n + \left( \frac{\partial p}{\partial t} \right)^{BD,n+\frac{1}{2}}
\]
Scale-dependent time integration

Scale-dependence via **multi-grid**

\[
p = \sum_{j=1}^{J} p^{(j)}
\]

where

\[
p^{(j)} = (1 - P \circ R) R^{j-1} p \quad \text{with} \quad R : \text{MG restriction} \quad P : \text{MG prolongation}
\]

scale-dependent blending

\[
u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2} \frac{\partial}{\partial x} p^{n+\frac{1}{2}}
\]

\[
\sum_j \eta^{(j)} p^{(j) n + \frac{1}{2}} = \sum_j \eta^{(j)} p^{(j) n} - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^{n + \frac{1}{2}} - \sum_j (1 - \eta^{(j)}) \frac{\Delta t}{2} \left( \frac{\partial p^{(j)}}{\partial t} \right)^{\text{BD,} n + \frac{1}{2}}
\]
Scale-dependent time integration

implicit midpoint

new scheme

BDF2
Scale-dependent time integration

- Implicit midpoint
- New scheme
- BDF2

![Graphs showing different schemes at t = 3](image-url)
Model Equations – Dispersion Relation and Amplitude

Implicit midpoint rule:

$CFL=1$

$CFL=10$

S. Vater & R. Klein (FU Berlin) Scale Dependent Discretizations PAKT Wolken Treffen 10 / 29
Model Equations – Dispersion Relation and Amplitude

BDF-2:
CFL=1

CFL=10
Dispersion Relation and Amplitude

Blended Scheme

CFL=10
Limit regimes in atmospheric flows

Sound-proof limits

Semi-implicit scheme for compressible flows

Scale-dependent time integration

Extensions: Moisture & general Eqs. of State
Moist pseudo-incompressible model

Bryan’s moist bubble test case

Run with straight pseudo-incompressible model*

Thermodynamically consistent version is work in progress
Conclusions
Publications


