Variational formulations of soundproof models

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Abstract

Goals of this talk:

1. Set up a variational framework for deriving ideal (nondissipative) soundproof fluid equations.
2. Derive the old soundproof models and a few new ones.
3. Discuss with you what should be done next.

1 Sound

The fastest-moving atmospheric and oceanic waves are the sound waves, whose presence can adversely affect numerical simulations of atmospheric and ocean circulations, by poisoning the desired low frequency circulations with high frequency oscillations. The numerical simulations can be made *soundproof* by replacing the exact governing equations with an approximate system that does not possess sound waves.

We will use standard variational methods to derive soundproof fluid models.

Sound waves in 1D obey the following PDE in the density (D) with $c^2 = [\partial p/\partial D]_\theta := const$

$$D_{tt} - c^2 D_{xx} = 0 \quad \text{for} \quad D(x,t) \in \mathbb{R}_+$$

By standard variational methods (Fermat [1650], Lagrange [1770]) this wave equation follows from

$$0 = \delta S = \delta \int_{t_0}^{t_1} L(D)\, dt = \int_{t_0}^{t_1} \left\langle \frac{\delta L}{\delta D}, \delta D \right\rangle dt$$

$$\text{choose } \delta \int_{t_0}^{t_1} L\, dt \quad = \delta \int_{t_0}^{t_1} \frac{1}{2}(D_t^2 - c^2 D_x^2)\, dx\, dt$$

$$= - \int_{t_0}^{t_1} (D_{tt} - c^2 D_{xx}) \delta D\, dx\, dt$$

for constant wave speeds $\pm c$.

Note: $\langle \cdot\, , \, \cdot \rangle$ above denotes L^2 pairing in 1D.

Sound proofing the 1D sound equation

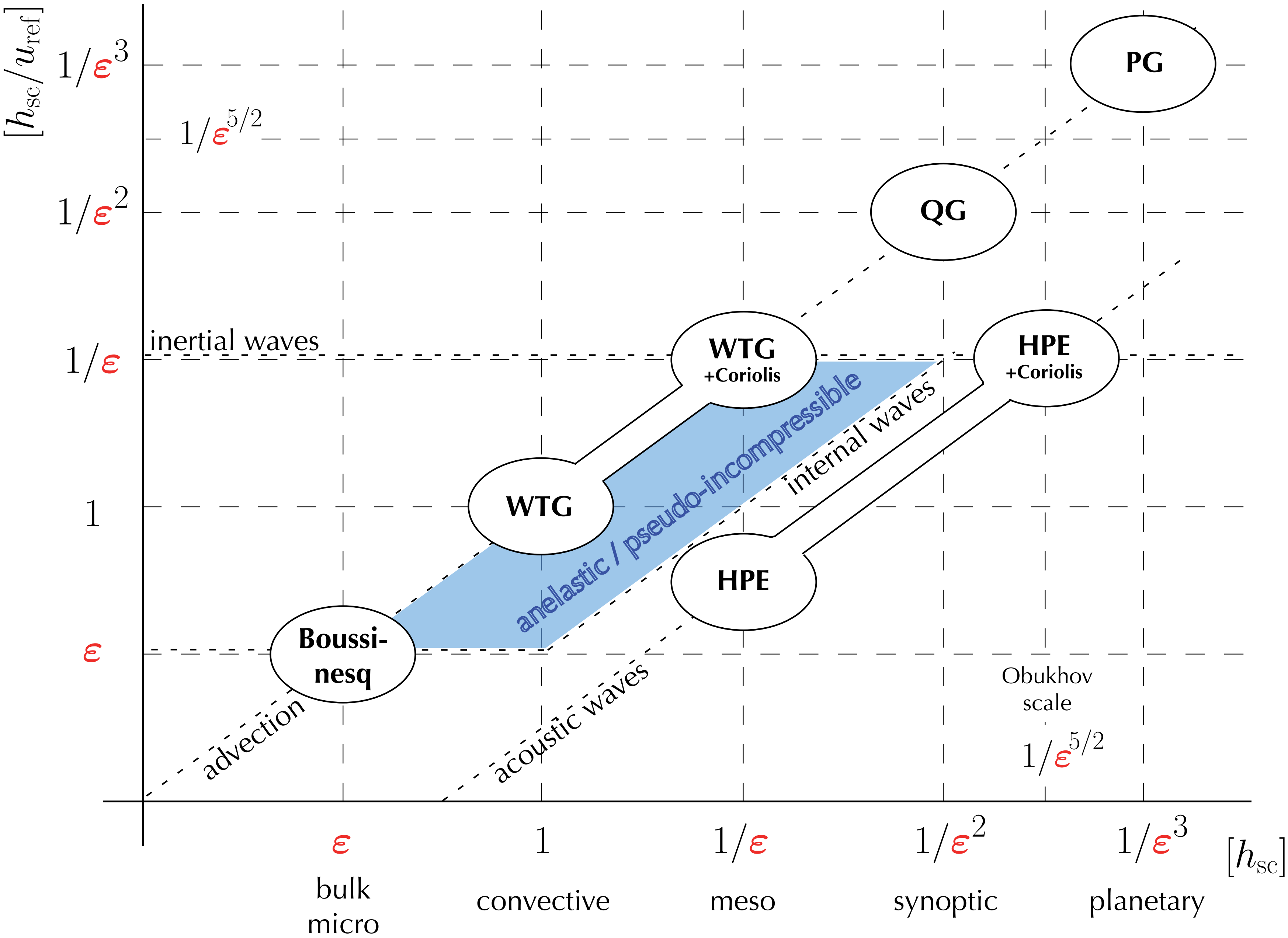
Try imposing the constraint $D = 1$ with Lagrange multiplier p. That is, set

$$0 = \delta S = \delta \int_{t_0}^{t_1} \frac{1}{2}(D_t^2 - c^2 D_x^2) + p(D-1)\,dx\,dt$$
$$= \int_{t_0}^{t_1} (D_{tt} - c^2 D_{xx} + p)\delta D + \delta p(D-1)\,dx\,dt$$

- Then, of course, no waves propagate, because D is constant.

- This is trivial in 1D, because there aren't enough variables to absorb the constraint sensibly.

- In a moment, though, we will do something similar but more meaningful for fluid dynamics.



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Variational formulation of ideal fluid dynamics

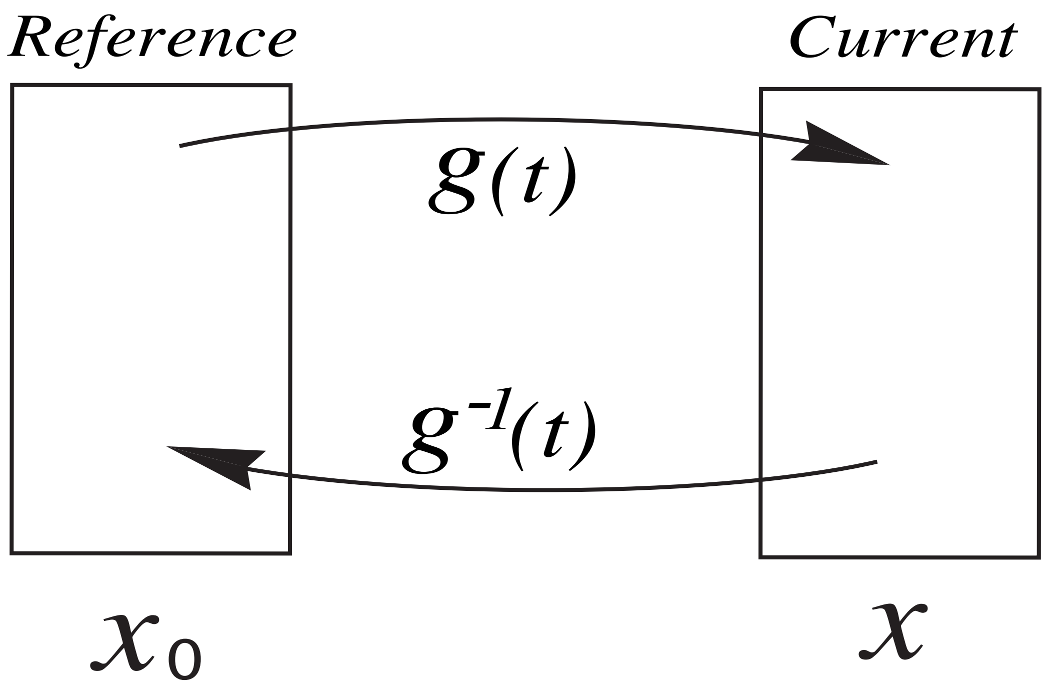
Theorem 1. The equations of ideal fluid dynamics follow from Hamilton's principle [HMR98, HMR02]

$$0 = \delta S = \delta \int_{t_0}^{t_1} \ell(u, a)\, dt = \int_{t_0}^{t_1} \left\langle \frac{\delta \ell}{\delta u}, \delta u \right\rangle + \left\langle \frac{\delta \ell}{\delta a}, \delta a \right\rangle dt$$

where u is the Eulerian fluid velocity and $a \in V$ represents the set of properties advected by the fluid.

Remark 2.

The key idea in using this HP is to represent the variations δu and δa in terms of Lagrangian particle paths, raised to paths in the Lie group $\mathrm{Diff}(\Omega)$ of smooth invertible maps of the domain Ω into itself.



Proof. The key idea. Let

$$x(t, x_0) = g_t x_0$$

where $g_t : \mathbb{R} \to \mathrm{Diff}(\Omega)$ and subscript t denotes dependence on time $t \in \mathbb{R}$.

If $x(t,x_0)=g_t x_0$, then

$$\dot{x}(t,x_0)=\dot{g}_t x_0 = u(x(t,x_0),t) = u_t\circ g_t x_0 \quad \text{and} \quad a_0(x_0)=a_t\circ g_t x_0$$
$$\Longrightarrow \quad u_t=\dot{g}_t g_t^{-1} \quad \text{and} \quad a_t=a_0 g_t^{-1}=g_t^* a \quad \Longrightarrow \quad (\partial_t+\pounds_u)a_t=0\,.$$

In the advection law $(\partial_t+\pounds_u)a_t$ denotes the partial time derivative of the advected quantity a_t, plus its Lie derivative along the vector field u.

$(\partial_t+\pounds_u)$ is the Eulerian expression for the total Lagrangian time derivative.

For example, when $a=D(x,t)d^n x$ we compute the continuity equation as

$$(\partial_t+\pounds_u)\big(D(x,t)d^n x\big)=\frac{d}{dt}\big(D(x(t,x_0),t)d^n x(t,x_0)\big) \quad \text{along} \quad \frac{dx}{dt}=u(x,t)$$
$$=\big(\partial_t D+\nabla D\cdot u+D\,\mathrm{div}u\big)\,d^n x$$

Lemma 3.
For the vector field $\eta = \delta g g^{-1} \in \mathfrak{X}(\Omega)$, the following variational formulas hold

$$\delta u = \partial_t \eta + \pounds_\eta u \quad \text{and} \quad \delta a = -\, \pounds_\eta a \,,$$

where $\pounds$ denotes Lie derivative. For example, $\delta a = \frac{d}{d\epsilon}\big|_{\epsilon=0} a_0 g_\epsilon^{-1} = -\pounds_\eta a$.

Proof. Write $u = \dot{g} g^{-1}$ and $\eta = \delta g g^{-1} = g' g^{-1}$ in natural notation and express the partial derivatives

$$\dot{g} = \partial g / \partial t \quad \text{and} \quad \delta g = g' = \partial g / \partial \epsilon|_{\epsilon=0}$$

by using the right translations as

$$\dot{g} = u \circ g \quad \text{and} \quad g' = \eta \circ g \,.$$

Then a simple use of the chain rule proves $\delta u = \partial_t \eta + \pounds_\eta u$.

Likewise, $\delta a = -\, \pounds_\eta a$ follows from the definition $a = a_0 g^{-1}$ by computing

$$\delta a = a_0 \delta(g^{-1}) = -\, a_0 g^{-1} \delta g g^{-1} = -\, a\eta = -\, \pounds_\eta a$$

Details of the proof of the Lemma:

By the chain rule, the mixed partial derivatives of these definitions satisfy

$$\dot{g}' = u' = \nabla u \cdot \eta \quad \text{and} \quad \dot{g}' = \dot{\eta} = \nabla \eta \cdot u \,.$$

The difference of the mixed partial derivatives implies the desired formula,

$$u' - \dot{\eta} = \nabla u \cdot \eta - \nabla \eta \cdot u = [\eta, u] =: \pounds_{\eta} u \,,$$

so that

$$u' = \dot{\eta} + \pounds_{\eta} u \,.$$

$\square$

Continuing the proof of the theorem:

By the Lemma, we have

$$\delta u = \partial_t \eta + \pounds_\eta u \quad \text{and} \quad \delta a = -\, \pounds_\eta a.$$

Consequently, we find

$$\begin{aligned}
0 = \delta S &= \int_{t_0}^{t_1} \left\langle \frac{\delta \ell}{\delta u}, \delta u \right\rangle + \left\langle \frac{\delta \ell}{\delta a}, \delta a \right\rangle \, dt \\
&= \int_{t_0}^{t_1} \left\langle \frac{\delta \ell}{\delta u}, \partial_t \eta + \pounds_\eta u \right\rangle_{\mathfrak{X}} + \left\langle \frac{\delta \ell}{\delta a}, -\, \pounds_\eta u \right\rangle_V \, dt \\
&= \int_{t_0}^{t_1} \left\langle -\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} - \frac{\delta \ell}{\delta u} \diamond u + \frac{\delta \ell}{\delta a} \diamond a \, , \, \eta \right\rangle_{\mathfrak{X}} \, dt
\end{aligned}$$

The operation diamond $(\diamond)$ is defined by equating two different pairings:

$$\Big\langle b \diamond a, \eta \Big\rangle_{\mathfrak{X}} := \Big\langle b, -\, \pounds_\eta a \Big\rangle_V \,,$$

where $a \in V$ and $b \in V^*$ are L^2 dual quantities.

Remark: Diamond $(\diamond)$ puts δu and δa on the same footing.

Conclusion of the proof

Hamilton's principle $\delta S = 0$ has yielded the Euler-Poincaré (EP) equation

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta u} \diamond u = \frac{\delta \ell}{\delta a} \diamond a \,,$$

which is completed by the auxiliary geometrical equation for the advected quantities a

$$(\partial_t + \pounds_u) a = 0 \,.$$

$\square$

Summary

Hamilton's principle has produced the following Euler-Poincaré (EP) system of equations

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta u} \diamond u = \frac{\delta \ell}{\delta a} \diamond a \quad \text{and} \quad (\partial_t + \pounds_u) a = 0 \,.$$

The diamond operation $(\diamond)$ must be calculated for each quantity, depending on how it transforms under $\mathrm{Diff}(\Omega)$

$$u : \text{vector field,} \quad \frac{\delta \ell}{\delta u} \diamond u = \pounds_u \frac{\delta \ell}{\delta u}$$

$$D : \text{mass density,} \quad \frac{\delta \ell}{\delta D} \diamond D = -\, D \nabla \frac{\delta \ell}{\delta D} \cdot dx \otimes d^n x = -\, d\left(\frac{\delta \ell}{\delta D}\right) \otimes D\, d^n x$$

$$\theta : \text{potential temperature,} \quad \frac{\delta \ell}{\delta \theta} \diamond \theta = \frac{\delta \ell}{\delta \theta} \nabla \theta \cdot dx \otimes d^n x = \frac{\delta \ell}{\delta \theta}\, d\theta \otimes d^n x$$

Outlook for the remainder of the talk

We now have the EP variational framework that we will need to derive soundproof fluid models.

First, though, we will prove
(1) Kelvin's circulation theorem
and
(2) Conservation of potential vorticity
in the EP variational framework

Kelvin circulation theorem.

The EP system is

$$(\partial_t + \pounds_u)\frac{\delta \ell}{\delta u} = \frac{\delta \ell}{\delta a} \diamond a \quad \text{and} \quad (\partial_t + \pounds_u) a = 0\,.$$

The corresponding Kelvin circulation theorem emerges on using

$$(\partial_t + \pounds_u) D = \partial_t D + \nabla \cdot (Du) = 0.$$

Then we calculate

$$\frac{d}{dt}\oint_{c(u)} \frac{1}{D}\frac{\delta \ell}{\delta u} = \oint_{c(u)} (\partial_t + \pounds_u)\left(\frac{1}{D}\frac{\delta \ell}{\delta u}\right) = \oint_{c(u)} \frac{1}{D}\frac{\delta \ell}{\delta a} \diamond a$$

$$\text{which for } (D,\theta) \in a = \oint_{c(u)} -d\left(\frac{\delta \ell}{\delta D}\right) + \frac{1}{D}\frac{\delta \ell}{\delta \theta}\, d\theta = \oint_{c(u)} \frac{1}{D}\frac{\delta \ell}{\delta \theta}\, d\theta$$

So Kelvin's theorem is,

$$\frac{d}{dt}\oint_{c(u)} \frac{1}{D}\frac{\delta \ell}{\delta u} = \oint_{c(u)} \frac{1}{D}\frac{\delta \ell}{\delta \theta}\, d\theta$$

Stokes theorem now gives us

$$(\partial_t + \pounds_u) d \left(\frac{1}{D} \frac{\delta \ell}{\delta u} \right) = d \left(\frac{1}{D} \frac{\delta \ell}{\delta \theta} \right) \wedge d\theta$$

Hence, on using $(\partial_t + \pounds_u)\theta = \partial_t \theta + u \cdot \nabla \theta = 0$ and

$$[d, \pounds_u] = d\pounds_u - \pounds_u d = 0\,,$$

we find

$$(\partial_t + \pounds_u) \left(d \left(\frac{1}{D} \frac{\delta \ell}{\delta u} \right) \wedge d\theta \right) = \left(d \left(\frac{1}{D} \frac{\delta \ell}{\delta \theta} \right) \wedge d\theta \right) \wedge d\theta = 0$$

Therefore, the scalar potential vorticity

$$q := D^{-1} \nabla \theta \cdot \mathrm{curl} \left(\frac{1}{D} \frac{\delta \ell}{\delta \mathbf{u}} \right)$$

is conserved on fluid particles,

$$(\partial_t + \pounds_u) q = \partial_t q + u \cdot \nabla q = 0\,.$$

Soundproofing

For $(D,\theta) \in a$ we will use Lagrange multiplier p to constrain the Lagrangian to relate the mass density D and potential temperature θ by

$$0 = \delta S = \delta \int_{t_0}^{t_1} \ell(u, D, \theta) + \underbrace{p\Big(D_0(z)\mathit{\Theta}(\theta_0(z)) - D\,\mathit{\Theta}(\theta)\Big)}_{\text{SP constraint}}\,dt$$

where $\Theta(\theta)$ is a function of the potential temperature to be chosen shortly.

Then the δp variation will yield

$$\delta p\Big(D_0(z)\mathit{\Theta}(\theta_0(z)) - D\Theta(\theta)\Big) = 0$$

This SP constraint ties D to θ and makes $D\Theta(\theta)$ time-independent.

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The product $D\Theta(\theta) = D_0(z)\,\Theta(\theta_0(z))$ is frozen into the particle motion,

$$\frac{\partial(D\Theta(\theta))}{\partial t} + \nabla\cdot(D\Theta(\theta)\mathbf{u}) = 0\,,$$

and its time-independence produces a type of incompressibility relation

$$\nabla\cdot\big(D_0(z)\,\Theta(\theta_0(z))\mathbf{u}\big) = 0\,,$$

whose preservation in time determines the pressure p in the motion equation.

Thus, the SP constraint prohibits waves from propagating through the fluid. In particular, no sound can propagate in a fluid model that obeys this constraint. In other words, this constraint produces soundproof models.

Next, we will write the EP equations for this constrained Lagrangian and choose $\Theta(\theta)$ so as to recover previous soundproof ideal fluid models.

2 Hamilton's principle for known soundproof GFD motion equations

For the soundproof (SP) models, we have proposed a Lagrangian given by

$$\ell_{\rm SP} = \int \bigg[D\Big(\frac{1}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - gz - C_v\pi_0(z)\,\theta\Big) + \; p\underbrace{\Big(D_0(z)\Theta(\theta_0(z)) - D\,\Theta(\theta)\Big)}_{\text{SP constraint}} \;\bigg] d^3x\,. \qquad (2.1)$$

Recall that D denotes the mass density, $\mathbf{u}$ is the Eulerian fluid velocity, $\mathbf{R}$ is a vector field whose curl is $2\mathbf{\Omega}$ (twice the local angular rotation vector), θ is the total potential temperature, $\theta_0(z)$, $\pi_0(z)$ and $D_0(z)$ are reference profiles, C_v is a constant and Θ is an arbitrary smooth function that we are free to choose.

Theorem 4. Let $\Theta(\theta) = 1 - \alpha + \alpha\theta$. Then the EP system (1) for the Lagrangian (2.1) recovers the following known SP models:

1. The anelastic approximation (AA) of [OP62], [LH82] and [Ban96] for $\alpha = 0$;
2. The pseudo-incompressible approximation (PIA) of [Dur89, Dur08] for $\alpha = 1$, and
3. The divergence-free (Euler) flows for $\alpha = 0$ and $D_0(z) = 1$.

Corollary 5. The quasi-hydrostatic versions of these theories follow by ignoring the kinetic energy of vertical motion in these Lagrangians, before taking variations. See Arakawa and Konor [2009] [AK09].

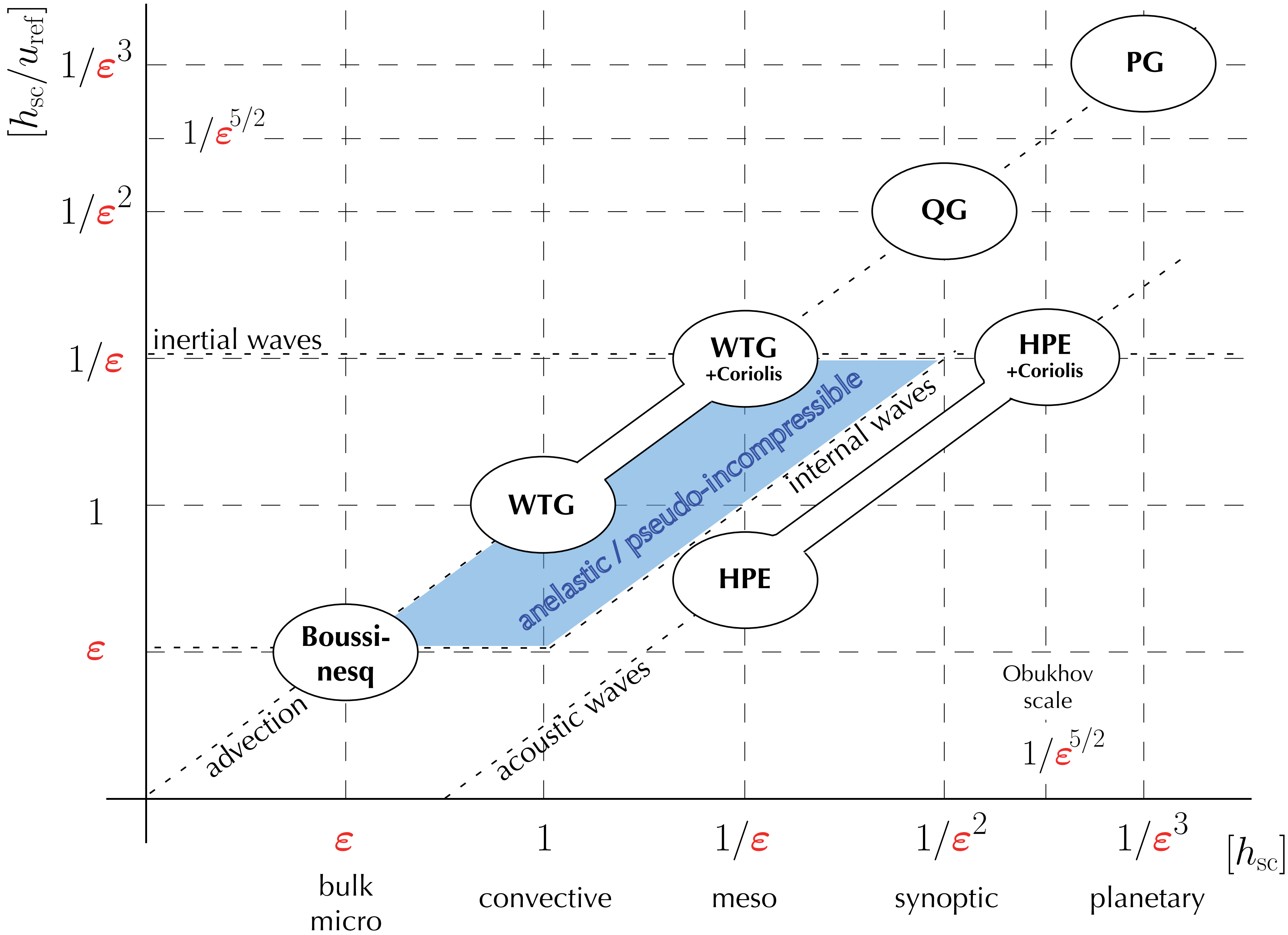

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Table 1: GFD models arise from asymptotics in Hamilton's principle.

$$l_{\text{Euler}} = \int \Big[D(1+b)\Big(\underbrace{\mathbf{R}(\mathbf{x})\cdot\mathbf{u}}_{\text{Rotation}} + \underbrace{\frac{\epsilon}{2}|\mathbf{u}|^2 + \frac{\epsilon}{2}\sigma^2 w^2}_{\text{Kinetic Energy}} - \underbrace{\frac{z}{\epsilon\mathcal{F}}}_{\text{Pot Energy}} \Big) - \underbrace{p(D-1)}_{\text{Constraint}} \Big] \mathrm{d}^3x$$

- $l_{\text{Euler}} \to l_{\text{EB}}$, for small buoyancy, $b = O(\epsilon)$.
- $l_{\text{EB}} \to l_{\text{PE}}$, for small aspect ratio, $\sigma^2 = O(\epsilon)$.
- $l_{\text{PE}} \to l_{\text{HBE}}$, for horizontal velocity decomposition,

$$\mathbf{u} = \hat{\boldsymbol{z}} \times \nabla\psi + \epsilon\nabla\chi = \mathbf{u}_R + \epsilon\mathbf{u}_D\,, \quad \text{and } |\mathbf{u}|^2 \to |\mathbf{u}_R|^2 \text{ in } l_{\text{PE}}\,.$$

- $l_{\text{HBE}} \to l_1$ [Sal88], for horizontal velocity,

$$\mathbf{u} = \mathbf{u}_1 = \hat{\boldsymbol{z}} \times \nabla\tilde{\phi}\,,$$

where

$$\tilde{\phi}(\mathbf{x}_3, t) = \phi_S(x, y, t) + \int_z^0 \mathrm{d}z'\, b\,,$$

i.e. $\partial\tilde{\phi}/\partial z = -\,b$ and dropping terms of order $O(\epsilon^2)$ in l_{HBE}.

- $l_1 \to l_{QG}$, on dropping terms of order $O(\epsilon^2)$ in the Euler–Poincaré equations for l_1.

Table 2: **Non-dimensional Lagrangians for GFD models.**

$$l_{\text{Euler}} = \int \Big[D(1+b)\Big(\mathbf{R}(\mathbf{x})\cdot\mathbf{u} + \frac{\epsilon}{2}|\mathbf{u}|^2 + \frac{\epsilon}{2}\sigma^2 w^2 - \frac{z}{\epsilon\mathcal{F}}\Big) - p(D-1)\Big]\,\mathrm{d}^3x$$

$$l_{\text{EB}} = \int \Big[D\Big(\mathbf{R}\cdot\mathbf{u} + \frac{\epsilon}{2}|\mathbf{u}|^2 + \frac{\epsilon}{2}\sigma^2 w^2 - bz\Big) - p(D-1)\Big]\,\mathrm{d}^3x$$

$$l_{\text{PE}} = \int \Big[D\Big(\mathbf{R}\cdot\mathbf{u} + \frac{\epsilon}{2}|\mathbf{u}|^2 - bz\Big) - p(D-1)\Big]\,\mathrm{d}^3x$$

$$l_{\text{HBE}} = \int \Big[D\Big(\mathbf{R}\cdot\mathbf{u} + \frac{\epsilon}{2}|\mathbf{u} - \epsilon\mathbf{u}_D|^2 - bz\Big) - p(D-1)\Big]\,\mathrm{d}^3x$$

$$l_1 = \int \Big[D\Big((\mathbf{R} + \epsilon\mathbf{u}_1)\cdot\mathbf{u} - \frac{\epsilon}{2}|\mathbf{u}_1|^2 - bz\Big) - p(D-1)\Big]\,\mathrm{d}^3x$$

$$l_{\text{QG}} = \int_{\mathcal{D}}\int_{z_0}^{z_1} \Big[D\Big(\mathbf{R}\cdot\mathbf{u} + \frac{\epsilon}{2}\mathbf{u}\cdot(1 - \mathcal{L}(z)\Delta^{-1})\mathbf{u}\Big) - p(D-1)\Big]\,\mathrm{d}z\,\mathrm{d}^2x\,,$$

where

$$\mathcal{L}(z) = \Big(\frac{\partial}{\partial z} + B\Big)\frac{1}{\mathcal{S}(z)}\Big(\frac{\partial}{\partial z} - B\Big) - \mathcal{F}$$

and $B = 0$ for standard QG.

Discussion Questions:

(1) Can we put dissipation into soundproof fluid models?
(Yes! Diffuse the momentum, $m = \delta\ell/\delta u$)

(2) Can we use this variational framework to develop numerical codes?
(Yes! Would it be worthwhile to do it? Probably!)

(3) Can we use the soundproof idea to try to keep weather prediction dynamics on a slow manifold? (Don't know. Any ideas?)

(4) Does soundproofing limit the wavenumber of rapid change in the solutions, say, due to convective over-turning and rapid adjustment?
(Hint: The primitive equations still allow very high wavenumbers.)

(5) Can we do multi-scale soundproofing to model the effects of seasonal variations on the local weather, say, by making the reference state time-dependent?
(Don't know. Any ideas?)

Discussion Questions, continued:

(6) For multi-scale soundproofing, should we consider large-scale flow as Lagrangian reference coordinates for small-scale flow?

(7) Are there other ways to make the model soundproof?

What about simply regularising to slow down the sound waves?

Would this be good enough?

(8) What other problem(s) should we be trying to solve using this variational framework?

(9) What is the exact relation of the EP variational framework to Rupert Klein's paper, Scale-Dependent Models for Atmospheric Flows, [Kle10]?

(10) What else needs to be done, in order of priority?

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