Efficient algorithms for non-linear four-wave interactions in discrete spectral wave models

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Purpose

- Provide an overview of developments and requirements in the development of efficient algorithms for the computation of non-linear four-wave interactions (S_{nl4}) in operational discrete spectral wave models

- Discuss the concept 'efficient algorithm' in relation to computational costs and wave model performance and types of application

Contents

- Importance of S_{nl4} in wave evolution
- Analytical methods
- Numerical methods
- Discrete Interaction Approximations
- Quasi-exact methods
- Shallow water aspects
- Conclusions

Importance of S_{nl4}

- Phillips (1960) showed basic principle of non-linear four-wave interaction
- Theory extended to random surface gravity waves by Hasselmann (1962) and Zakharov (1968)
- JONSWAP experiment (1973) concluded that S_{nl4} is mainly responsible for forward shift of peak frequency
- Shape stabilization and influence on spectral shape, both in frequency and direction space
- Small and large time scale interactions

Basic equation of S_{nl4}

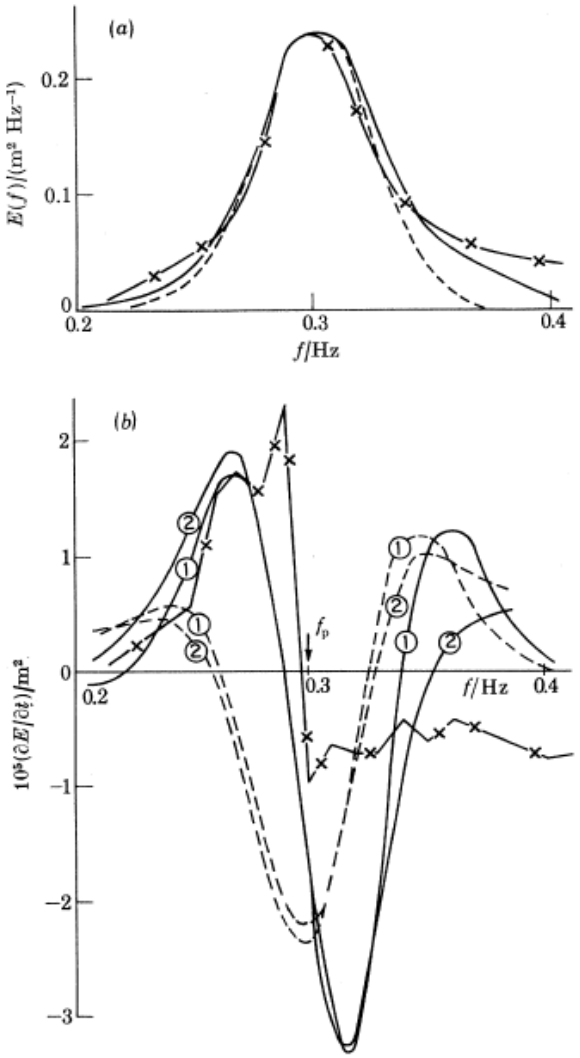
- Rate of change of action density in wavenumber $\boldsymbol{k}_1$ function of four wave numbers involved in a resonant interaction

$$\frac{\partial n_1}{\partial t} = \iiint G\left(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4\right) \times \delta\left(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4\right) \times \delta\left(\omega_1 + \omega_2 - \omega_3 - \omega_4\right)$$
$$\times \left[n_1 n_3 \left(n_4 - n_2\right) + n_1 n_4 \left(n_3 - n_1\right) \right] d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4$$

- Six-fold integral over the wave numbers $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$
- Delta functions reflect resonance conditions and ensure conservation of wave energy, action and momentum
- G complicated coupling coefficient, scales with k^6
- On a basic level triple products of action densities involved $n_i\, n_j\, n_k$

Computational methods for Snl4

- Narrow peak approximations Longuet-Higgins (1975), Fox (1978)
- Finite width approximation Dungey & Hui (1979)
- Provided valuable insights into nature of S_{nl4}
- Narrow peak approximations suffer from insufficient degrees of freedom to be applicable in discrete spectral wave models (only broad scale)

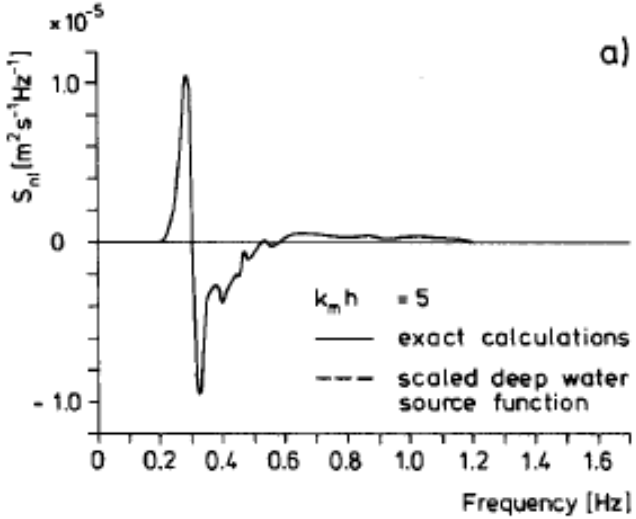


Analytical methods

- Rewrite transfer integral to eliminate delta-functions and to make transfer integral computationally feasible
- At least three basic analytical transformations exist in literature
- Webb (1978) - Masuda (1980) - Lavrenov (2001)
- Methods differ in various ways:
  - choice of integration variables, i.e. Webb uses $\boldsymbol{k}_1$ and $\boldsymbol{k}_3$, Masuda uses $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$
  - treatment of singularities
  - internal transformations and approximations

Numerical methods

- Sell and Hasselmann (1962) pioneers to evaluate transfer integral numerically
- Symmetric method developed by Hasselmann and Hasselmann (1981) leading to EXACT-NL model
- Ragged behaviour of EXACT-NL related to different spectral resolutions of input spectrum and computational grid



- Webb -> WRT method: Tracy, Resio, Perrie, Van Vledder
  Masuda -> RIAM method: Komatsu, Hashimoto
  Lavrenov -> GQM method: Benoit, Gagnaire-Renou

Performance numerical methods

- WRT – RIAM – GQM: Like their analytical masters these computational methods differ in various (hidden, unknown) ways, but do they provide the same answers? They are often used as reference in development of approximate methods.

- Which of these computational methods is the most efficient?

- Inter-comparison study for S_{nl4} is now being carried out (Van Vledder, Benoit, Hashimoto, Resio, Tolman, …) to asses methods

- Comparison includes:
  - Individual spectra and aspects as spectral resolution, spectral shape, directional properties, symmetries, integration methods, depth effects, integration ranges, assumptions
  - Dynamic wave model performance in combination with other source terms

Hamburg developments at MPI

- Exact-NL model first 3G discrete spectral 1d (time or fetch) wave model: $\partial N/\partial x$ or $\partial N/\partial T = S = X_{nl} + S_{wind} + S_{wcap}$

- Exact-NL used for studies of source term balance, e.g.:
  - Komen, Hasselmann & Hasselmann (1984) searching for fully developed wave spectra
  - Finite depth gravity waves (Weber, 1988)
  - Directional response in turning wind fields (Van Vledder & Holthuijsen, 1993)

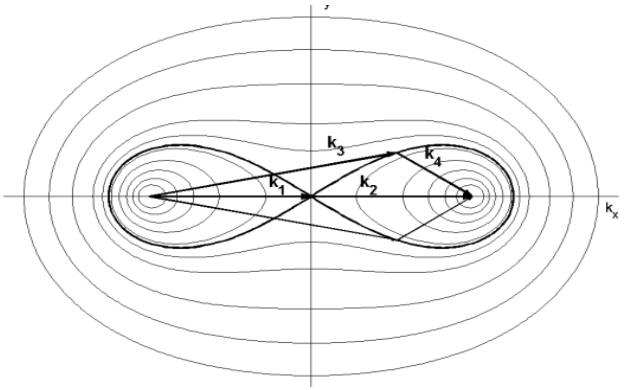
- Computational methods too time consuming (not efficient) for application in operational discrete spectra 2D wave models

- Development of efficient Discrete Interaction Approximation and leading to development of the WAM model (and Wavewatch, SWAN, CREST, WWM, …)

Discrete Interaction Approximation (DIA)

- Discrete Interaction Approximation (DIA) evaluates **one** subset of all possible interaction configurations with λ=0.25

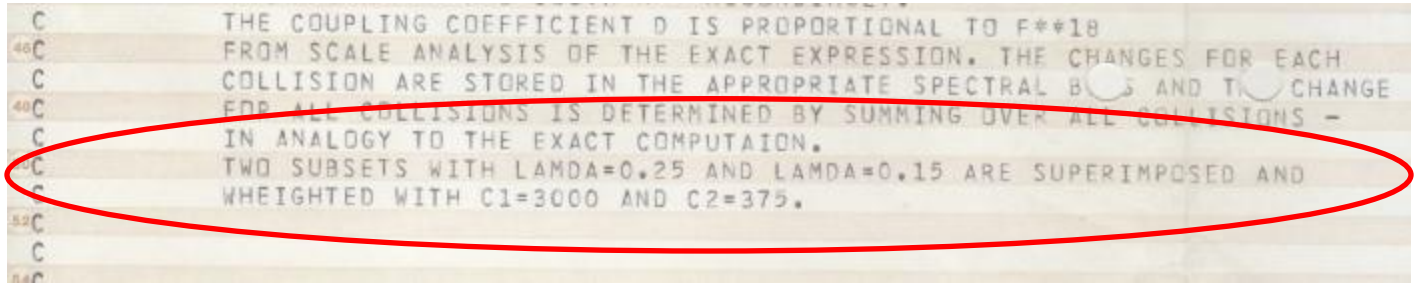
- $$\boldsymbol{k}_1 = \boldsymbol{k}_2$$
$$\sigma_1 = \sigma_2 = \sigma$$
$$\sigma_3 = (1+\lambda)\sigma$$
$$\sigma_4 = (1-\lambda)\sigma$$



- DIA quite successful, but its deficiencies became gradually troublesome as they hamper development of other source terms, Van Vledder et al. (2000)

- Errors in DIA are usually 'corrected' by tuning of other source terms

Original DIA

- The original Discrete Interaction Approximation (DIA) of Hasselmann had two configurations: λ_1=0.25 and λ_1=0.15 with weights of 3000 and 375
- 
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C     FOR ALL COLLISIONS IS DETERMINED BY SUMMING OVER ALL COLLISIONS -
C     IN ANALOGY TO THE EXACT COMPUTAION.
C     TWO SUBSETS WITH LAMDA=0.25 AND LAMDA=0.15 ARE SUPERIMPOSED AND
C     WHEIGHTED WITH C1=3000 AND C2=375.
C
C
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- The second configuration was dropped because it's added value in terms of wave model performance was insufficient with respect to model **efficiency** !

Extension of the DIA

- Adding additional λ-configurations
  - Van Vledder et al. (2000), 2 configurations
  - Hashimoto & Kawaguchi (2001), up to 5 configurations
- Multiple λ-configurations have insufficient degrees of freedom to fully represent full nonlinear transfer.
- When n_λ=5, exhaustion is reached in the number of unique triplets (products of $N_i \times N_j \times N_k$), Van Vledder (2005)

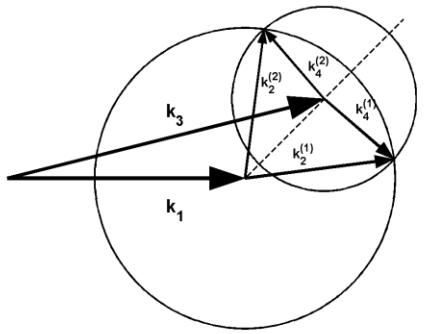
Generalized Multiple DIA

- Generalized DIA with arbitrary configuration proposed by Van Vledder (2001); cast in symmetric form by Tolman (2003)
- MDIA in principle able to represent full transfer using multiple configurations
- Final GMD (Tolman, 2011):

$$\boldsymbol{k}_1 \neq \boldsymbol{k}_2$$
$$\theta_2 = \theta_1 + \Delta\theta$$
$$\sigma_1 = \sigma$$
$$\sigma_2 = (1+\mu)\sigma$$
$$\sigma_3 = (1+\lambda)\sigma$$
$$\sigma_4 = (1-\mu-\lambda)\sigma$$

$$\boldsymbol{k}_1 \neq \boldsymbol{k}_2$$
$$\theta_2 = \theta_1 + \Delta\theta$$
$$\sigma_1 = (1+\lambda)\sigma$$
$$\sigma_2 = (1-\lambda)\sigma$$
$$\sigma_3 = (1+\mu)\sigma$$
$$\sigma_4 = (1-\mu)\sigma$$

$$\begin{pmatrix} \delta S_{nl,1} \\ \delta S_{nl,2} \\ \delta S_{nl,3} \\ \delta S_{nl,4} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \left(\frac{1}{n_d} C_{deep} B_{deep} + \frac{1}{n_s} C_{shal} B_{shal} \right) \times$$
$$\times \left[\left(\frac{c_{g1}E_1}{\sigma_1 k_1} \right) \left(\frac{c_{g2}E_2}{\sigma_2 k_2} \right) \left\{ \left(\frac{c_{g3}E_3}{\sigma_3 k_3} \right) + \left(\frac{c_{g4}E_4}{\sigma_4 k_4} \right) \right\} - \left\{ \left(\frac{c_{g1}E_1}{\sigma_1 k_1} \right) + \left(\frac{c_{g2}E_2}{\sigma_2 k_2} \right) \right\} \left(\frac{c_{g3}E_3}{\sigma_3 k_3} \right) \left(\frac{c_{g4}E_4}{\sigma_4 k_4} \right) \right]$$



Determining coefficients of MDIA

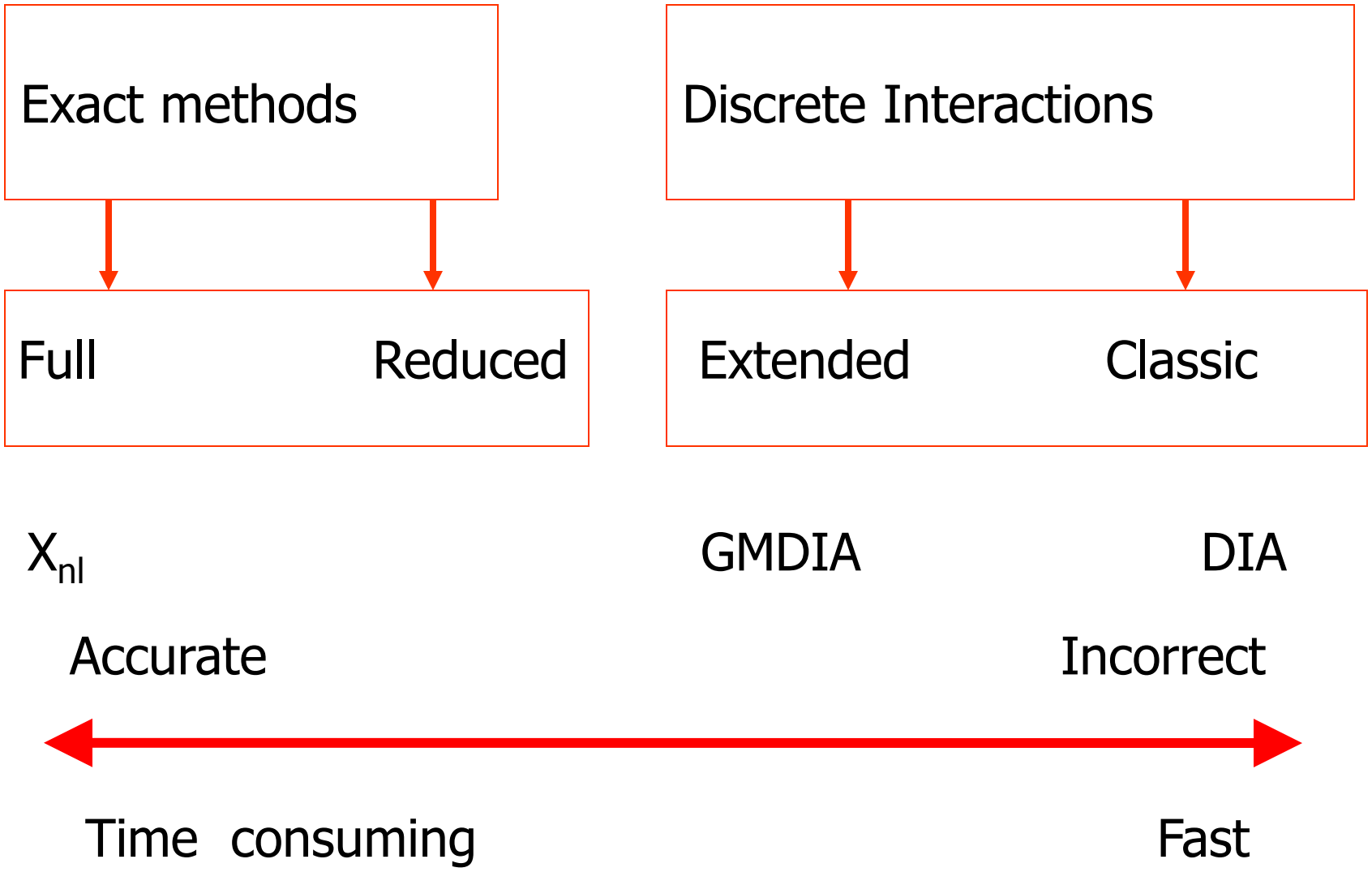
- First attempts to determine coefficients of a multiple DIA were based on optimization against a limited set of (academic) test spectra using least-square analysis or error mapping, while using X_{nl} results as reference
- Process is time consuming when number of configurations increases
- What is a good set of test spectra? Is it representative?
- What is next best set of configuration when extending the DIA?
- No guarantee that MDIA provides good model performance
- Verify efficient approximations in wide range of full model runs against range of parameters H_{m0}, T_p, T_{m02}, T_{m01}, $T_{m-1,0}$, θ, σ, κ, ...

Holistic optimization of a (Generalized Multiple) DIA

- Hasselmann et al. (1985) chose only one configuration and λ=0.25 in view of model efficiency and ability to reproduce growth curves
- Tolman and co-workers (2003-2010) tried error mapping procedures with varying amounts of success
- Tolman (2010) applies holistic optimization and a genetic algorithm to choose shapes and weights of individual configurations of GMD's
- Holistic in view of a large set of academic and fields cases representing many possible realistic cases

Results of optimized GMD

- Results obtained by Tolman (2010) very good. It is a major improvement over the classic DIA
- Various GMD configurations are proposed with various degrees of complexity and related computational requirements
- Error measure reduced by 60% for a GMD setting with 5 configurations
- Tolman (2010) used X_{nl} based on WRT method as ground truth
- Determination of optimal configurations time consuming, $O(10^5)$ non-stationary model runs
- Optimal GMD configuration(s) depend on choice of other source terms, characteristics of host model, spectral resolution, spatial discretization and set of model runs


Exact methods
Discrete Interactions
Full
Reduced
Extended
Classic
X_{nl}
GMDIA
DIA
Accurate
Incorrect
Time consuming
Fast

Bridging the gap between GMD and Xnl

- GMD of Tolman (2012) shows good model performance against reasonable costs (subjective criterion)
- Number of unique configurations still O(10^3) lower than X_{nl} based methods
- This mismatch suggests that almost all quadruplets involved in the evaluation of X_{nl} do not significantly contribute to the transfer integral, but which ?
- Reduced versions of X_{nl} being developed (quasi-exact approaches) by filtering, higher order integration techniques, smarter interpolation, smart assumptions, …

Quasi-exact methods

- Two-scale approximation (TSA) of Resio and Perrie (2009, 2010) distinguishes broad scale and local scale.
  - Broad scale computed exactly with WRT for pre-selected spectra
  - Local correction of residues
  - Limited applicability for complex spectra

- Advanced Dominant Interaction transfer approximation of Perrie, Susilo and Toulany (2010), by selecting part of transfer integral contributing most to total transfer rate. Poor performance in operational forecasts

- SRIAM (Komatsu & Masuda); kind of MDIA with about 20 configurations. Good performance in operational conditions. Still costly

- Diffusion approximations (Zakharov, Pushkarev)

- Neural Networks (Tolman & Krasnopolsky)

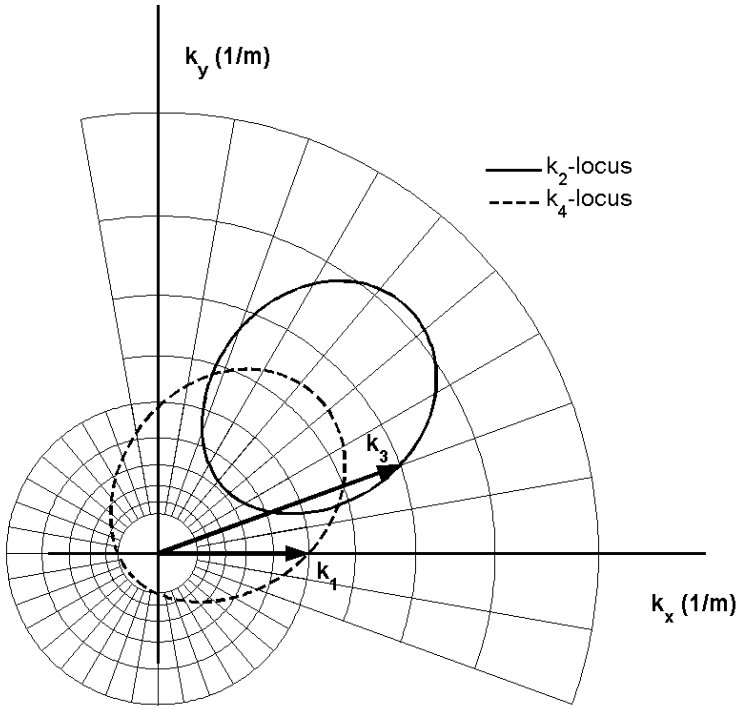
Reducing exact method to mimic a Discrete Interaction

- Reducing WRT using mathematically consistent reduction of integration space. Can be considered as a top-down approach
- Workhorse is the WRT method (but a similar methodology may apply to the RIAM and GQM methods)

$$\frac{\partial \boldsymbol{n}_1}{\partial t} = \int d\boldsymbol{k}_3 T\left(\boldsymbol{k}_1, \boldsymbol{k}_3\right)$$

$$T\left(\boldsymbol{k}_1, \boldsymbol{k}_3\right) = \int_s ds G(s) J(s) N_p(s)$$

The T-function in the WRT method



$\boldsymbol{k}_1$ and $\boldsymbol{k}_3$ loop over all discrete wave numbers of a spectrum

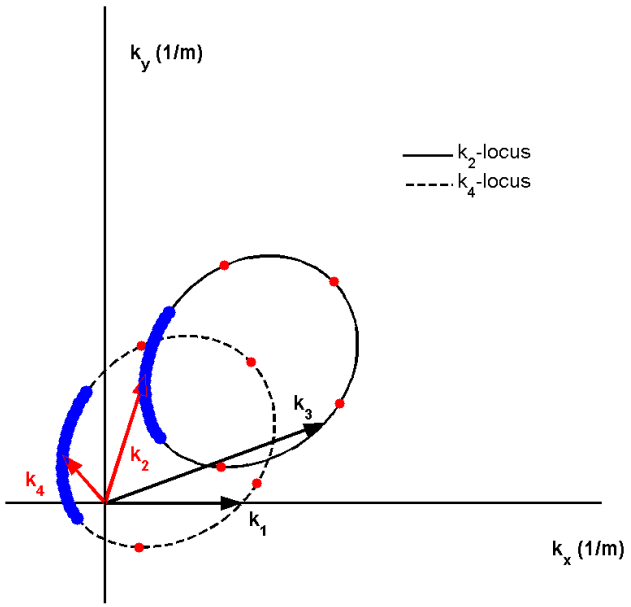
For each $\boldsymbol{k}_1,\boldsymbol{k}_3$ combination the resonant $\boldsymbol{k}_2$ and $\boldsymbol{k}_4$ wave numbers form closed path s (locus)

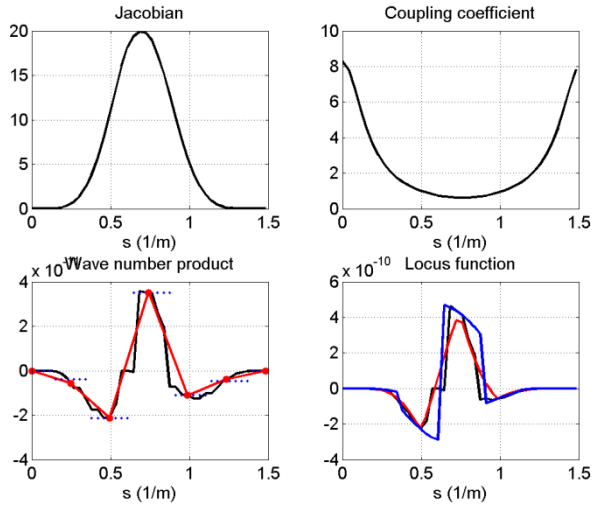
$T(\boldsymbol{k}_1,\boldsymbol{k}_3)$ integrates product of functions along locus; coupling coefficient $G(s)$, Jacobian term $J(s)$, wave number product $N_p(s)$

Bi-linear interpolation of nearest bin to evaluate locus function, option to save time

Integration along locus, LQA method

Pick a few points on locus, but keep all information of G and J

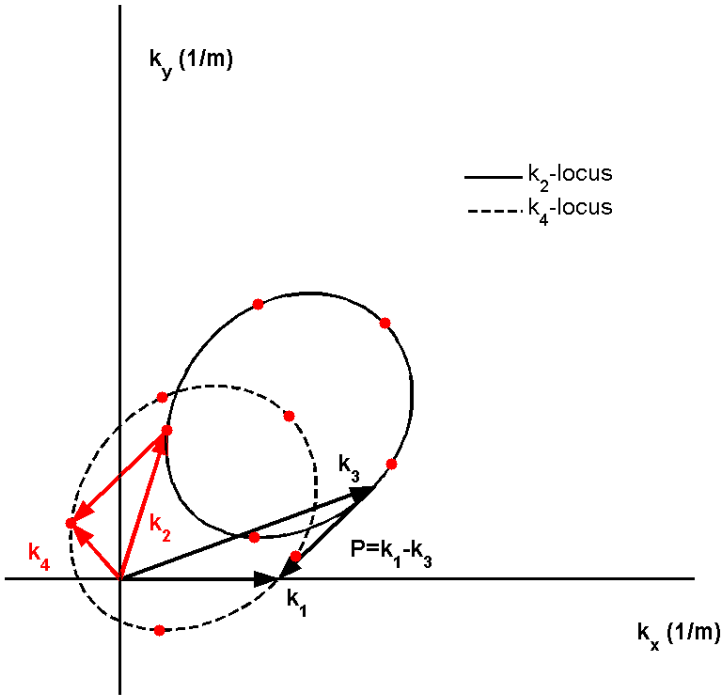




Piece wise integration along locus, lump contribution of coupling coefficient G and Jacobian J, which can be precomputed

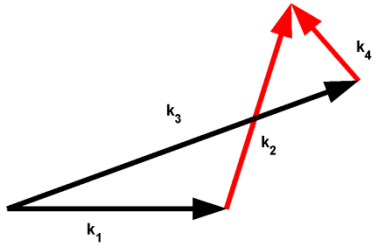
$$T = \sum_{i=1}^{N} N(s_i) \int_{s_i - 0.5\Delta s_i}^{s_i + 0.5\Delta s_i} G(s) J(s) ds$$

Incremental integration along locus



Dual points on locus with $\boldsymbol{k}_1$ and $\boldsymbol{k}_3$ form a quadruplet

Identify individual wave number configurations on locus



Determine shape factors λ, μ, $\Delta\theta$ for each quadruplet

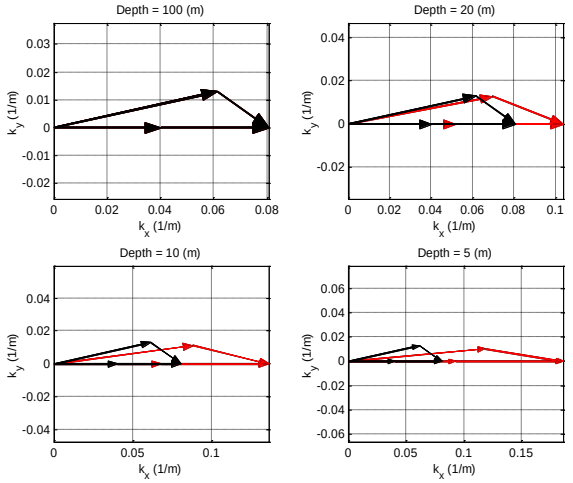
Generate set of discrete interactions with associated weights

Equivalence of reduced WRT and Discrete Interaction

- In WRT changes are made **only** to each pair of discrete $n(\boldsymbol{k}_1)$ and $n(\boldsymbol{k}_3)$, while using information from loci of $\boldsymbol{k}_2$ and $\boldsymbol{k}_4$. Action densities at the latter wave numbers are affected further on in the looping process.
- In DIA changes are made simultaneously to all four wave numbers in a configuration of $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$
- Principle of detailed balance $\Delta n_1=\Delta n_2=-\Delta n_3=-\Delta n_4$ connects methods
- Strength of individual $\boldsymbol{T}$-contributions determine weights of quadruplets. Account for scaling with wave number
- Good results obtained with reduced LQA versions. Mismatch still $O(10^2)$

Shallow water aspects

- Basic principles of finding an efficient algorithm are equal for deep and shallow water
- In shallow water shape of configuration depends on depth
- Additional data storage and handling of pre-computed interaction coefficients, Jacobeans and interpolation weights
- Theoretical developments in coupling coefficient in intermediate depths (Janssen and Onorato, 2007), where transfer rate goes to 0 for kh = 1.363, implications under investigation
- Theoretical developments by Stiassnie & Gramstad (2012) about validity of S_{nl4} in non-homogenous situations
- Not (yet) related to topic of efficiency

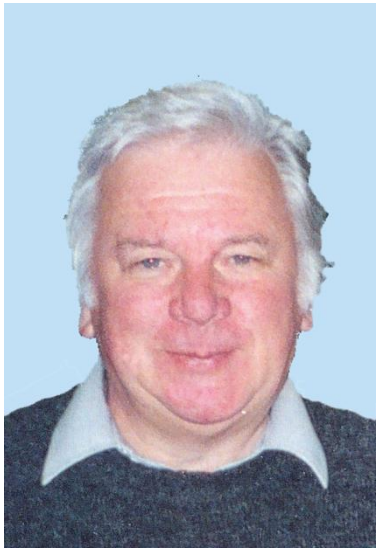


Conclusions

- The concept 'efficient algorithm' for S_{nl4} must be viewed in relation to model performance and computational requirements
- … not only against its ability to efficiently approximate X_{nl}
- Efficiency should also be considered in relation to types of model application
- Model validation should include H_{m0}, T-measures, mean direction θ, directional spreading σ, spectral narrowness κ, ….
- GMD is a (good) bottom-up approach but (maybe slightly) limited due to its dependence on choice of other source terms and model settings
- Reduced X_{nl} top-down approach may provide efficient generic solution

Questions ?

Klaus Hasselmann



Vladimir Zakharov