Some issues in numerical stochastic weather/climate modeling

or

How do I use Stochastic Differential Equations to model something real?

Cécile Penland, with thanks to Prashant Sardeshmukh, Roger Témam, Brian Ewald, James A. Hansen, and many others.
Outline

• Schematic
• A Theorem
• Stochastic Taylor expansion (leading to)
• Some integration schemes

Examples, including cautionary tales

• Some ways to beat the system
• List of useful references
Let’s say we have a simple system:

\[
\frac{dx}{dt} = G(x) + F(t)
\]
But our simple system may not be so simple at a finer timescale:

\[ \frac{dx}{dt} = G(x) + F(t) \]
Choose a scaling $s = \varepsilon^2 t$:

$$\frac{dx}{ds} = G(x, s/\varepsilon^2) + \frac{1}{\varepsilon} F(x, s/\varepsilon^2) \quad (*)$$

For simplicity, say

$$F_i(x, s/\varepsilon^2) = \sum_k F^k_i(x, s) \eta_k (s/\varepsilon^2)$$

and

$$C_{km} = \int_{-\infty}^{\infty} <\eta_k(t)\eta_m(t'+t) > \, dt' \equiv (\Phi\Phi^T)_{km}$$

Lim $(*) \to \begin{align*}
\lim_{t \to \infty} \lim_{\varepsilon \to 0} \frac{dx}{ds} &= G(x, s) \, ds + \sum_{k, \alpha} \sum_k F^k_i(x, s) \phi_{k\alpha} \cdot dW_\alpha \\
(W \text{ is a Brownian motion; } dW \in \mathcal{N}(0, dt))
\end{align*}$
Wiener process, or Brownian motion

- \( \langle W(t) \rangle = 0 \)
- \( \langle W(t)W(t') \rangle = \min(t,t') \)
- \( \langle dW(t)dW(t') \rangle = dt \delta(t-t') \)
- Two sets of integration rules found in nature.

\[
\int_0^t W \, dW = W^2(t) - \frac{t}{2}
\]

\[
\int_0^t W \, dW = W^2(t)
\]
And now: Back to Calculus 101!

Deterministic system:

\[
\frac{dX(t)}{dt} = f(X) \\
\frac{dF(X)}{dt} = \frac{dF}{dX} \frac{dX(t)}{dt} = f(X) \frac{dF}{dX}
\]

Just rewriting:

\[
dF(X) = LF(X)dt, \quad L = f(X) \frac{d}{dX}
\]

Through an iterative procedure we get

\[
F(X(T)) = F(X(0)) + LF(X(0))T + \frac{1}{2} L^2 F(X(0))T^2 + \ldots
\]
The stochastic version:

\[ dX = f(X,t)dt + g(X,t)(\bullet)dW. \]

Define:

\[ L^0 = \frac{\partial}{\partial t} + f(X,t) \frac{\partial}{\partial X} \]  

(Stratonovich)

\[ L^0 = \frac{\partial}{\partial t} + f(X,t) \frac{\partial}{\partial X} + g^2(X,t) \frac{1}{2} \frac{\partial^2}{\partial X^2} \]  

(Ito)

And for either case:

\[ L^1 = g(X,t) \frac{\partial}{\partial X} \]
Get the stochastic Taylor expansion by iterating

\[ F(X(T), T) = F(X(0), 0) + \int_{0}^{T} L^{1} F(X(t), t)(\bullet) dW + \int_{0}^{T} L^{0} F(X(t), t) dt \]

We develop integration schemes by identifying \( F(X) = X(t) \).

**Rules of order:** Depends on whether of system is to be integrated in sense of Ito or Stratonovich.

**Ito:** Count 1 for every integral over time and 1/2 for every integral over dW, unless term has only time (i.e., no dW) in it. In that case, subtract 1/2 from what it would otherwise be.

**Stratonovich:** only schemes of total integer orders are valid.
So now we want to integrate

\[ dX = f(X,t)dt + g(X,t)(\bullet)dW. \]

**Lowest order (0.5): Stochastic Euler scheme:** \((\mathcal{R} \in \mathcal{N}(0,1))\).

\[ X(t_{n+1}) = X(t_n) + f(X_n,t) \Delta + g(X_n,t)(\Delta W), \]

where \(\Delta W = \mathcal{R} \sqrt{\Delta} \).
Time for a cautionary tale:

\[ \frac{dX(t)}{dt} = -\gamma X(t) + \sigma \xi \]

Let’s compare:

\[ X(t_{n+1}) = X(t_n) - \gamma X(t_n) \Delta + \sigma \mathcal{R} \sqrt{\Delta} \] (Euler)

\[ X(t_{n+1}) = X(t_n) + (-\gamma X(t_n) + \sigma \mathcal{R}) \Delta \] (Naïve Euler)
Again we want to integrate
\[ dX = f(X,t)dt + g(X,t)(\bullet)dW. \]

This time we’ll use an explicit \( O(1) \) Mil’steyn scheme: \( (\mathcal{N} \in \mathcal{N}(0,1)) \).

\[
X(t_{n+1}) = X(t_n) + f(X,t_n)\Delta + g(X,t_n)\Delta W
+ g(X,t_n)\frac{\partial g(X,t_n)}{\partial X} I_{(1,1)} \quad \Delta W = \mathcal{N} \sqrt{\Delta}.
\]

Stratonovich: \( I_{(1,1)} = (\Delta W)^2/2 \)

Ito: \( I_{(1,1)} = [(\Delta W)^2 - \Delta]/2 \)

Comments: Multivariate system more complicated; use same variate; etc.
Time for another cautionary tale:

\[ dX(t) = L X(t) dt + S dW \]

\[
\langle XX^T \rangle = \begin{pmatrix}
2 & 0 & 1 & 0 \\
0 & 2 & 0 & 1 \\
1 & 0 & 2 & 0 \\
0 & 1 & 0 & 2
\end{pmatrix}
\]

do i = 1,n
   r(i)=gasdev(idm)*sqrt(dt)
enddo

do i = 1,n
   x(i)=x0(i)
   do j = 1,n
      x(i)=x(i)+L(i,j)*dt
      +S(i,j)*r(j)
   enddo
enddo

do i = 1,n
   x(i)=x(i)+L(i,j)*dt
   +S(i,j)*gasdev(idm)*sqrt(dt)
enddo
enddo
Time for another cautionary tale:

\[ dX(t) = LX(t)dt + SdW \]

\[
<XX^T> = \begin{pmatrix}
2 & 0 & 1 & 0 \\
0 & 2 & 0 & 1 \\
1 & 0 & 2 & 0 \\
0 & 1 & 0 & 2
\end{pmatrix}
\]

\[
\text{do } i = 1, n \\
r(i) = \text{gasdev(idm)} \times \text{sqrt(dt)} \\
\text{enddo}
\]

do i = 1, n
\[ x(i) = x0(i) \]
do j = 1, n
\[ x(i) = x(i) + L(i, j) \times dt \]

\[ + S(i, j) \times r(j) \]
.enddo
.enddo

\[
<XX^T> = \begin{pmatrix}
1.98 & -0.01 & 0.99 & 0.00 \\
-0.01 & 1.99 & 0.02 & 1.00 \\
0.99 & 0.02 & 2.02 & -0.01 \\
0.00 & 1.00 & -0.01 & 2.01
\end{pmatrix}
\]

do i = 1, n
\[ x(i) = x(i) + L(i, j) \times dt \]
do j = 1, n
\[ x(i) = x(i) + \text{gasdev(idm)} \times \text{sqrt(dt)} \]
\[ + S(i, j) \times r(j) \]
.enddo
.enddo
Time for another cautionary tale:

\[ dX(t) = LX(t)dt + SdW \]

\[
<XX^T> = \begin{pmatrix}
2 & 0 & 1 & 0 \\
0 & 2 & 0 & 1 \\
1 & 0 & 2 & 0 \\
0 & 1 & 0 & 2
\end{pmatrix}
\]

\[
<XX^T> = \begin{pmatrix}
1.98 & -0.01 & 0.99 & 0.00 \\
-0.01 & 1.99 & 0.02 & 1.00 \\
0.99 & 0.02 & 2.02 & -0.01 \\
0.00 & 1.00 & -0.01 & 2.01
\end{pmatrix}
\]

\[
<XX^T> = \begin{pmatrix}
2.00 & 0.01 & -0.01 & 0.00 \\
0.01 & 2.02 & 0.01 & -0.01 \\
-0.01 & 0.01 & 2.01 & 0.00 \\
0.00 & -0.01 & 0.00 & 2.02
\end{pmatrix}
\]

do i = 1,n
\[
r(i) = \text{gasdev}(idm) \times \sqrt{dt}
\]
enddo

do i = 1,n
\[
x(i) = x0(i)
\]
do j = 1,n
\[
x(i) = x(i) + L(i,j) \times dt \\
+ S(i,j) \times r(j)
\]
enddo
enddo

\[
<XX^T> = \begin{pmatrix}
2.00 & 0.01 & -0.01 & 0.00 \\
0.01 & 2.02 & 0.01 & -0.01 \\
-0.01 & 0.01 & 2.01 & 0.00 \\
0.00 & -0.01 & 0.00 & 2.02
\end{pmatrix}
\]
What about implicit or semi-implicit schemes?

- Implicit Euler
- Implicit Mil’steyn
- Implicit Ewald-Témam
What about implicit or semi-implicit schemes?

- Implicit Euler
- Implicit Mil’steyn
- Implicit Ewald-Témam

*Never, ever, put a random number into the denominator!*
Yet another cautionary tale:

\[
\frac{dx}{dt} = (k^2 - r)x + F \quad \quad r = r_o + \eta \xi, \quad \xi \text{ stochastic}
\]

Heavy line: Analytic solution

Dots: Implicit Ewald-Témam

Light line: Naïve implicit scheme:

\[
x'(t+2\Delta) = x(t) + 2F\Delta
\]

\[
x(t+2\Delta) = x'(t+2\Delta) + 2\Delta [k^2 - (r_o + \eta\xi)] x(t+2\Delta)
\]

\[
x(t+2\Delta) = [x(t) + 2F\Delta] / \{1 - 2\Delta [k^2 - (r_o + \eta\xi)]\}
\]

To 7.8
This looks like a lot of work! Can’t I just stochastify a few parameters without rewriting the whole MODEL???
This looks like a lot of work! Can’t I just stochastify a few parameters without rewriting the whole MODEL???

Sometimes.
Can’t I just stochastify a few parameters without rewriting the whole model?

In explicit schemes such as Runge-Kutta, you can often get away with using deterministic code by replacing the amplitude $\sigma$ of the noise with $\sigma/\sqrt{\Delta}$.

If your timestep $\Delta$ is so small that it’s smaller than any dynamical timescale in your system, you may not have to inject the randomness every timestep. If I inject the noise every $N$ timesteps, I might get away with deterministic code by replacing the amplitude $\sigma$ of the noise with $\sigma\sqrt{N\Delta}$. Useful if $\sigma$ is diagnosed through data assimilation with assimilation window $= N\Delta$.

Caveat Emptor!
Some References
(Also see references therein.)

• Ewald, B., C. Penland, and R. Téمام 2004: Accurate Integration of Stochastic Climate Models. *Monthly Weather Review*, 132, 154-164. (Beware of Fig. 1.)