Mass-conserving and positive-definite semi-Lagrangian advection in NCEP GFS: 

*Decomposition for massively parallel computing without halo*

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Introduction

• The common elements for traditional semi-Lagrangian method are
  – Iteration to find departure and/or mid-point values
  – Interpolation from regular grid points to departure and/or mid-points
  – Require halo grids in MPP

• Advantage of semi-Lagrangian Method
  – Allowable larger time step, saving time
  – Use linear grid in spectral model; same grid-point with higher resolution of spectral truncation
\[ \frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} + v \frac{\partial q}{\partial y} = 0 \]

\[ D_q \frac{q_{n+1}}{Dt} = 0 \]

\[ q_{A_{n+1}} = q_{D_{n-1}} \]
Starting from mid-point

No guessing and no iteration

but one 2-D interpolation and one 2-D remapping
Proposed Method

- Splitting semi-Lagrangian advection
  - advection in one direction first
  - then advection in another direction
  - temporal and spatial splitting

- Advantage
  - no guessing and no iteration
  - 1-D interpolation and remapping
  - possible no halo (with transpose)
  - incremental implementation

- Possible to add mass conservation and positive definite advection
\[
\frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} + v \frac{\partial q}{\partial y} = 0
\]

\[
\frac{\partial q}{\partial t} \bigg|_{\text{X-direction}} + \frac{\partial q}{\partial t} \bigg|_{\text{Y-direction}} + u \frac{\partial q}{\partial x} + v \frac{\partial q}{\partial y} = 0
\]

\[
\left( \frac{\partial q}{\partial t} \right)_{\text{X-direction}} + u \frac{\partial q}{\partial x} + \left( \frac{\partial q}{\partial t} \right)_{\text{Y-direction}} + v \frac{\partial q}{\partial y} = 0
\]

\[
\left( \frac{Dq}{Dt} \right)_{\text{X-direction}} + \left( \frac{Dq}{Dt} \right)_{\text{Y-direction}} = 0
\]
\[ X_D = X_M - U_M \Delta t \]
\[ X_A = X_M + U_M \Delta t \]
\[ q_A^{n+1} = q_D^{n-1} \]

Interpolation \(q_D^{n-1}\)  
relocation \(q_A^{n+1}\)

remapping \(q^{n+1}\)
\[ Y_D = Y_M - V_M \Delta t \]
\[ Y_A = Y_M + V_M \Delta t \]
\[ q^{n+1}_A = q^{*n+1}_D \]
Gaussian 256 x 128 with time step of 1800 sec
Gaussian 256 x 128 with time step of 1800 sec
Across north pole
Gaussian 256 x 128 with time step of 1800 sec
Across south pole
at north pole

3day(color) analytical(black) 256x128 1800 sec
at equator
at south pole

9 day (color) analytical (black) 256x128 1800 sec
one complete revolution

12 day (color) analytical (black) 256x128 1800 sec
\[
L_1 = \frac{I[|q - q_T|]}{I[|q_T|]}
\]

\[
L_\infty = \frac{\text{Max}[|q - q_T|]}{\text{Max}[|q_T|]}
\]

\[
\text{max} = \frac{\text{Max}[q] - \text{Max}[q_T]}{\text{Max}[q_T] - \text{Min}[q_T]}
\]

\[
\text{min} = \frac{\text{Min}[q] - \text{Min}[q_T]}{\text{Max}[q_T] - \text{Min}[q_T]}
\]

\[
I[A] = \frac{1}{4\pi a^2} \int \int_{\phi \lambda} A \cos \phi d\phi d\lambda
\]

where

\[
\text{Max}[A] = \text{global}_\text{max}_\text{of}_A
\]

\[
\text{Min}[A] = \text{global}_\text{min}_\text{of}_A
\]
### Compare Errors

<table>
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<tr>
<th></th>
<th>min</th>
<th>max</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_{\infty}$</th>
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For mass conservation, let’s start from continuity equation

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho \zeta}{\partial \zeta} = 0 \]

Consider 1-D and rewrite it in advection form, we have

\[ \left( \frac{\partial \rho}{\partial t} \right)_{(X-direction)} + u \frac{\partial \rho}{\partial x} = -\rho \frac{\partial u}{\partial x} \]

Advection form is for semi-Lagrangian, but it is not conserved if divergence is treated as force at mid-point, So divergence term should be treated with advection
Divergence term in Lagrangian sense is the change of the volume if mass is conserved, so we can write divergence form as

$$\left(\frac{\partial u}{\partial x}\right)_{\text{Lagrangian sense}} = \frac{1}{\Delta x} \frac{d\Delta x}{dt}$$

Put it into the previous continuity equation, we have

$$\left(\frac{d\rho\Delta_x}{dt}\right)_{X\text{-direction}} = 0$$

$$\left(\frac{\partial\rho\Delta_x}{\partial t}\right)_{X\text{-direction}} + u \frac{\partial\rho\Delta_x}{\partial x} = 0$$

which can be seen as

$$\left(\rho\Delta_x\right)_{\text{departure}} = \left(\rho\Delta_x\right)_{\text{arrival}}$$
How about mass conservation for tracer?

If we use tracer and continuity equation as following together

\[ \frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} + v \frac{\partial q}{\partial y} + \zeta \frac{\partial q}{\partial \zeta} = 0 \]

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho \zeta}{\partial \zeta} = 0 \]

Then density weighted tracer can be treated as conservation as

\[ \frac{\partial \rho q}{\partial t} + \frac{\partial \rho q u}{\partial x} + \frac{\partial \rho q v}{\partial y} + \frac{\partial \rho q \zeta}{\partial \zeta} = 0 \]

Combine it with continuity equation, we can have conserved tracer advection

\[ \frac{d \rho q \Delta}{dt} = 0 \]

\[ \frac{d \rho \Delta}{dt} = 0 \]
We do

\[
X_L^D = X_L^M - U^M_L \Delta t \\
D
\]

\[
X_R^D = X_R^M - U^M_R \Delta t \\
D
\]

\[
\Delta_D = X_R^D - X_L^D \\
D
\]

\[
X_L^A = X_L^M + U^M_L \Delta t \\
A
\]

\[
X_R^A = X_R^M + U^M_R \Delta t \\
A
\]

\[
\Delta_A = X_R^A - X_L^A \\
A
\]

Interpolation \( \rho_{D}^{n-1} \)

\[
\rho_{D}^{n-1} \Delta_D = \rho_{A}^{n+1} \Delta_A \\
\rho
\]

relocation \( \rho_{A}^{*n+1} \)
\[ \rho_{D}^{n-1} \Delta_{D} = \rho_{A}^{n+1} \Delta_{A} \]
The given value can be presented piece-wisely by

\[ \rho = S(x) \]

so the previous mass equality can be replaced as following

\[ \int_{D_L}^{D_R} S_D^{n-1}(x) dx = \int_{A_L}^{A_R} S_A^{n+1}(x) dx \]

Also we want to make sure that total mass is conserved as

\[ \oint S_R^{n-1}(x) dx = \oint S_D^{n-1}(x) dx = \oint S_A^{n+1}(x) dx = \oint S_R^{n+1}(x) dx \]

where subscript R is regular grid
D is departure grid
A is arrival grid for

This implies that mass conservation should be used during interpolation from regular cell to departure cell and from arrival cell to regular cell. thus, we apply **monotonic PPM** for \( S(x) \).
\[
\frac{\text{(total mass) } - \text{(initial total mass)}}{\text{(initial total mass)}} \approx 10^{-15}
\]
Isochronal flow

- Any given point will return to its original location after a given period of time.
- Gaussian grid dimensioned 512 x 256.
- Rotate coordinates so “equator” goes through (39N,77W), thus giving flow over real poles.
- Apply non-divergent global wave number 4-20 perturbation displacement (of standard deviation size 0.1-0.2 non-dimensional vorticity) using a random number generator.
- Set return period to 10 days.
arbitrary tracer at initial condition 512x256
arbitrary tracer after 10 days 512x256 dt=900s
Decomposition in NCEP GFS

- Current NCEP GFS uses 1D decomposition with MPI and thread with OpenMP.
- Transpose with MPI_AllToAllv is used to move between two sub-domains for spectral transform.
- First sub-domain has some given latitudes with all longitudes grids, which is for FFT in longitudinal direction.
- Second sub-domain has some given zonal wave numbers with all meridian wave numbers, which is for Legendre transform in meridian direction.
- Transpose between two sub-domains, thus there is no halo required.
Implement into NCEP GFS

- The same first sub-domain is used to compute semi-Lagrangian advection in any given latitudinal global circle. All departure and arrival points are in the same circle, so no halo is needed.

- Then transpose first sub-domain to another grid-point sub-domain, which has all Gaussian latitude points but some longitude points. Therefore semi-Lagrangian advection can be computed in any given longitude with all latitude points, so, again, no halo is required.

- The PPM mass conserving between reduced grid and full grid in any given latitude is also applied.
Transpose

No halo is required
No increasing memory with
Increasing number of PE (cpu)
Halo Exchange

Extra memory is required, which may be as huge as computing grid while number of MPP cpu increases.
Case test in NCEP GFS

- Arbitrary date is selected.
- Modify NCEP GFS IO into grid-point data.
- Negative tracers are replaced with zero value at the initial time.
- T126 L64 resolution is tested.
- Model physics is included.
- Two runs are compared;
  - **control**: Spectral advection in horizontal, finite difference in vertical as operational GFS.
  - **nislfv**: Non-iteration mass conserving positive definite semi-Lagrangian on tracers.
November 4, 2008

13th on the use of HPC in meteorology

control

06h fcst specific humidity at model layer 40

nislfv
control

12h fcst specific humidity at model layer 40

nislfv

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control

24h fcst specific humidity at model layer 40

nislfv

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control

72h fcst specific humidity at model layer 40

nislfv
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6hr fcst cloud water at model layer 35

nislfv
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12hr fcst cloud water at model layer 30

control

nislfv
24hr fcst cloud water at model layer 30

nislfv

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control

72hr fcst cloud water at model layer 5

nislfv

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control

6hr fcst precipitation

nislfv
November 4, 2008

control

24hr fcst precipitation

nislfv

Precip(in/day) hour 24 with control

Precip(in/day) hour 24 with nislfv
control

72hr fcst precipitation

nislfv

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Conclusion & Future Work

- Modified traditional semi-Lagrangian without iteration to locate mid-/departure-points, but require interpolation and remapping with temporal and spatial split computation.
- Mass conserving is included with consideration of semi-Lagrangian for divergence.
- Positive definite is applied with monotone piecewise parabolic method (PPM) for interpolation/remapping.
- Due to spatial split, no halo is required since all required data for computation are all in the partial domain through transpose. Since no halo, there is no extra memory request, but it may have more data in communication than the method with halo and small number of cpu.
- Implement all prognostic variable, not only tracers, to have larger model time step to save integration cost.