Coarse Grained Stochastic Models for Tropical Convection and Climate

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1. Introduction

The current practical models for prediction of both weather and climate involve general circulation models (GCM) where the physical equations for these extremely complex flows are discretized in space and time and the effects of unresolved processes are parametrized according to various recipes. With the current generation of supercomputers, the smallest possible mesh spacings are about 50-100 km for short-term weather simulations and of order 200-300 km for short term climate simulations. There are many important physical processes which are unresolved in such simulations such as the mesoscale sea-ice cover, the cloud cover in sub-tropical boundary layers, and deep convective clouds in the tropics. An appealing way to represent these unresolved features is through a suitable coarse-grained stochastic model which simultaneously retains crucial physical features of the interaction between the unresolved and resolved scales in a GCM. In work from 2002 and 2003, the authors both have developed a new systematic stochastic strategy (Majda and Khouider 2002, Khouider et al. 2003) to parametrize key features of deep convection in the tropics involving suitable stochastic spin-flip models and also a systematic mathematical strategy to coarse-grain such microscopic stochastic models to practical mesoscopic meshes in a computationally efficient manner while retaining crucial physical properties of the interaction (Majda and Khouider 2002, Khouider et al. 2003). This last work is based on new strategies to systematically coarse grain stochastic lattice models which have achieved a computational speed up of order one billion in various materials science applications (Katsoulakis et al. 2003a, Katsoulakis et al. 2003b, Katsoulakis and Vlachos 2003). These procedures are tested on prototype deterministic/stochastic lattice models with a variety of dynamic bifurcations and phase transitions in recent work (Katsoulakis et al. 2003a, Katsoulakis et al. 2003b, Katsoulakis and Vlachos 2003).

As regards tropical convection, crucial scientific issues involve the fashion in which a stochastic model effects the climate mean state and the strength and nature of fluctuations about the climate mean. Here the strategy to develop a new family of coarse-grained stochastic models for tropical deep convection is briefly reviewed (Majda and Khouider 2002, Khouider et al. 2003). In (Khouider et al. 2003), it has been established that in suitable regimes of parameters, the coarse grained stochastic parametrizations can significantly alter the climatology as well as increase wave fluctuations about the climatology. This was established in (Khouider et al. 2003) in the simplest scenario for tropical climate involving the Walker circulation, the east-west climatological state which arises from local region of enhanced surface heat flux, mimicking the Indonesian marine continent.

2. The Microscopic Stochastic Model for CIN

In a typical GCM, the fluid dynamical and thermodynamical variables, denoted here by the generic vector, $\vec{u}$, are regarded as known only over a discrete horizontal mesh with $\vec{u}_j = \vec{u}(j\Delta x, t)$ denoting these discrete values. Throughout the discussion, one horizontal spatial dimension along the equator in the east-west direction is assumed for simplicity in notation and explanation. As mentioned above, the typical mesh spacing in a GCM is coarse with Δx ranging from 50 km to 250 km depending on the time duration of the simulation. The stochastic variable used to illustrate the approach is convective inhibition. Observationally, convective inhibition (CIN) is known to have significant fluctuations on a horizontal spatial scale on the order of a kilometer, the microscopic scale here, with changes in CIN attributed to different mechanisms in the turbulent boundary layer such as gust fronts, gravity waves, and turbulent fluctuations in equivalent potential temperature (Mapes 2000). In (Khouider et al. 2003), it was proposed that all of these different microscopic physical mechanisms and their interaction which increase and decrease CIN are too complex to model in detail in a coarse mesh GCM parametrization and instead, as in statistical mechanics, should be modeled by a simple order parameter, σ_I, taking only two discrete values,

$$\begin{aligned} \sigma_I &= 1 \qquad \text{at a site if convection is inhibited (a CIN site)} \\ \sigma_I &= 0 \qquad \text{at a site if there is potential for deep convection (a PAC site).} \end{aligned} \tag{1}$$

The value of CIN at a given coarse mesh point is determined by the averaging of CIN over the microscopic states in the vicinity of the given mesh point, i.e.,

$$\bar{\sigma}_I(j\Delta x, t) = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \sigma_I(x,t)\,dx. \tag{2}$$

Note that the mesh size, Δx, is mesoscopic, i.e., between the microscale, $O(1\text{ km})$, and the macroscale, $O(10,000\text{ km})$, and that $\bar{\sigma}_I$ can have any value in the range $0 \le \bar{\sigma}_I \le 1$. Discrete sums over microscopic mesh values (of order 1 km) and continuous integrals are utilized interchangeably for notational convenience.

As discussed in (Khouider et al. 2003), the microscopic CIN sites interact with each other and with the external mesoscopic variables, $\vec{u}_j$, through a set of plausible interaction rules. These rules are summarized through the microscopic energy for CIN in the boundary layer given by

$$H_h(\sigma_I) = \frac{1}{2}\sum_{x\neq y} J(x-y)\sigma_I(x)\sigma_I(y) + h_{\text{ext}}\sum_x \sigma_I(x) \tag{3}$$

where J is a symmetric interaction potential and h_{ext} is an external potential. Note that the microscopic energy is a monotonic increasing function of the external field h_{ext}. The boundary layer states are regarded as a heat bath coupled to the mesoscopic variables $\vec{u}_j$ via the external potential h_{ext} so that the equilibrium statistics are given by the Gibbs measure

$$G(\sigma) = \frac{1}{Z_\Lambda} e^{\beta H_h(\sigma)} d\sigma \tag{4}$$

For the microscopic dynamics, a configuration randomly flips at a site x,

$$\sigma_I^x(y) = \begin{cases} 1-\sigma_I(y), \text{ if } y = x \\ \sigma_I(y) \text{ if } y \neq x \end{cases} \tag{5}$$

as a jump Markov process where the rate $c(\sigma,x)$ is given by the Arrhenius adsorption/desorption model

$$c(\sigma,x) = \begin{cases} \frac{1}{\tau_I}e^{-\beta V(x)}, \text{ if } \sigma_I(x)=1 \\ \frac{1}{\tau_I} \text{ if } \sigma_I(x)=0 \end{cases} \tag{6}$$

and for which $G(\sigma)$ is the invariant measure (Katsoulakis et al. 2003b), with

$$V(x) = \sum_{z\neq x} J(x-z)\sigma_I(z) + h_{\text{ext}}. \tag{7}$$

Here τ_I is the characteristic interaction time.

3. The Simplest Coarse-Grained Stochastic Model for CIN

In practical parametrization, it is desirable for computational feasibility to replace the microscopic dynamics by a process on the coarse mesh which retains critical dynamical features of the interaction. Following the general procedure developed and tested in (Katsoulakis et al. 2003a, Katsoulakis et al. 2003b, Katsoulakis and Vlachos 2003) the simplest local version of the systematic coarse grained stochastic process is developed in (Khouider et al. 2003) and summarized here.

Each coarse cell Δx_k, $k=1,\cdots,m$, of the coarse-grained lattice is divided onto l microscopic cells such that $\Delta x_k \longleftrightarrow \frac{1}{l}\{1,2,\cdots,l\}, k=1,\cdots,m$. In the coarse-grained procedure, given the coarse-grained sequence of random variables

$$\eta_t(k) = \sum_{y\in\Delta x_k} \sigma_{I,t}(y), \tag{8}$$

so that the average in (2) verifies $\bar{\sigma}_I(j\Delta x) = \eta(k)/l$, for $j=k$ in some sense, the microscopic dynamics is replaced by a birth/death Markov process defined on the variables, $\{0,1,\cdots,l\}$, for each k such that $\eta_t(k)$ evolves according to the following probability law.

$$\begin{aligned}
&\text{Prob}\Big\{\eta_{t+\Delta t}(k) = n+1 \,|\, \eta_t(k)=n\Big\} = C_a(k,n)\Delta t + o(\Delta t)\\
&\text{Prob}\Big\{\eta_{t+\Delta t}(k) = n-1 \,|\, \eta_t(k)=n\Big\} = C_d(k,n)\Delta t + o(\Delta t)\\
&\text{Prob}\Big\{\eta_{t+\Delta t}(k) = n \,|\, \eta_t(k)=n\Big\} = 1-\Big(C_a(k,n)+C_d(k,n)\Big)\Delta t + o(\Delta t)\\
&\text{Prob}\Big\{\eta_{t+\Delta t}(k) \neq n, n-1, n+1 \,|\, \eta_t(k)=n\Big\} = o(\Delta t).
\end{aligned} \tag{9}$$

The coarse grained adsorption/desorption rates are given respectively by

$$\begin{aligned}
C_a(k,\eta) &= \frac{1}{\tau_I}[l-\eta(k)]\\
C_d(k,\eta) &= \frac{1}{\tau_I}\eta(k)e^{-\beta\bar{V}(k)}
\end{aligned} \tag{10}$$

where

$$\bar{V}(k) = \bar{J}(0,0)\big(\eta(k)-1\big) + h_{\text{ext}} \tag{11}$$

with the coarse grained interaction potential within the coarse cell given by $\bar{J}(0,0) = 2U_0/(l-1)$ where U_0 is the mean strength of the potential J (Katsoulakis et al. 2003a, Katsoulakis et al. 2003b). The coarse-grained energy content for CIN is given by the coarse-grained Hamiltonian

$$\bar{H}(\eta) = \frac{U_0}{l-1}\sum_k \eta(k)\big(\eta(k)-1\big) + h_{\text{ext}}\sum_k \eta(k). \tag{12}$$

The canonical invariant Gibbs measure for the coarse-grained stochastic process is a product measure given by

$$G_{m,l,\beta}(\eta) = (Z_{m,l,\beta})^{-1} e^{\beta \bar{H}(\eta)} P_{m,l}(d\eta) \tag{13}$$

where $P_{m,l}(d\eta)$ is an explicit prior distribution (Katsoulakis et al. 2003b). As shown in (Katsoulakis et al. 2003b), the coarse-grained birth/death process above satisfies detailed balance with respect to the Gibbs measure in (12) as well as a number of other attractive theoretical features. The simplest coarse-grained approximation given above assumes that the effect of the microscopic interactions on the mesoscopic scales occurs within the mesoscopic coarse-mesh scale, Δx, otherwise systematic nonlocal couplings are needed (Katsoulakis et al. 2003b). The accuracy of these approximations is tested for diverse examples from material science elsewhere (Katsoulakis et al. 2003a, Katsoulakis et al. 2003b, Katsoulakis and Vlachos 2003) and instructive idealized coupled models (Katsoulakis et al. 2004, Katsoulakis et al. 2005a, Katsoulakis and Vlachos 2005b).

The practical implementation of the coarse-grained birth/death process in (8)–(11) requires specification of the parameters, τ_I, U_0, q and the external potential $h_{\text{ext}}(\vec{u}_j)$ as well as the statistical parameter β.

4. The Model Deterministic Convective Parametrization

A prototype mass flux parametrization with crude vertical resolution (Majda and Shefter 2001, Majda et al. 2004) is utilized to illustrate the fashion in which the coarse-grained stochastic model for CIN can be coupled to a non-stochastic convective mass flux parametrization. The prognostic variables $(u, \theta, \theta_{eb}, \theta_{em})$ are the x-component of the fluid velocity, u, the potential temperature in the middle troposphere, θ, the equivalent potential temperatures, θ_{eb} and θ_{em}, measuring, respectively, the potential temperatures plus moisture content of the boundary layer and middle troposphere. The vertical structure is determined by projection on a first baroclinic heating mode (Majda and Shefter 2001, Majda et al. 2004). The dynamic equations for these variables in the parametrization are given by

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \bar{\bar{\alpha}} \frac{\partial \theta}{\partial x} = -\left(C_D^0 \frac{1}{h} \sqrt{u_0^2 + u^2} \right) u - \frac{1}{\tau_D} u \\
&\frac{\partial \theta}{\partial t} - \bar{\alpha} \frac{\partial u}{\partial x} = \mathscr{S} + Q_R^0 - \frac{\theta}{\tau_R} \\
&h \frac{\partial \theta_{eb}}{\partial t} = -D(\theta_{eb} - \theta_{em}) + \left(C_\theta \sqrt{u_0^2 + u^2} \right) (\theta_{eb}^* - \theta_{eb}) \\
&H \frac{\partial \theta_{em}}{\partial t} = D(\theta_{eb} - \theta_{em}) + Q_R^0 - \frac{\theta_{em}}{\tau_R}
\end{aligned} \tag{14}$$

while the constants Q_R^0, θ_{eb}^* are externally imposed and represent the radiative cooling at equilibrium in the upper troposphere and saturation equivalent potential temperature in the boundary layer. The constants h and H measure the depths of the boundary layer and the troposphere above the boundary layer, respectively. The typical values used here are $h = 500$ m and $H = 16$ km while $u_0 = 2$ m s^{-1}. The explicit values for the other constants used in (14) and elsewhere in this section can be found in (Majda and Shefter 2001, Majda et al. 2004).

The vertically integrated equivalent potential temperature given by

$$\langle \theta_e \rangle_z = \frac{1}{H+h} [h\theta_{eb} + H\theta_{em}] \approx \frac{h}{H} \theta_{eb} + \theta_{em}$$

satisfies the conservation equation

$$\frac{\partial \langle \theta_e \rangle_z}{\partial t} = \left(\frac{C_\theta^0}{H} \sqrt{u_0^2 + u^2} \right) (\theta_{eb}^* - \theta_{eb}) + Q_R^0 - \frac{\theta_{em}}{\tau_R}. \tag{15}$$

That is $\langle \theta_e \rangle_z$ is conserved in the absence of surface evaporative heating and tropospheric radiative cooling. The crucial quantities in the prototype mass flux parametrization are the terms $\mathscr{S}$ and D where $\mathscr{S}$ represents the middle troposphere heating due to deep convection while D represents the downward mass flux on the boundary layer. The heating term $\mathscr{S}$ is given by

$$\mathscr{S} = M\sigma_c \big((\mathrm{CAPE})^+ \big)^{1/2} \tag{16}$$

with M a fixed constant, σ_c the area fraction for deep convective mass flux, and CAPE $= \theta_{eb} - \gamma\theta$, the convectively available potential energy. The downward mass flux on the boundary layer, D, includes the environmental downdrafts, m_e, and the downward mass flux due to convection, m_-, which are non-negative quantities so that

$$\begin{aligned} &D = m_e + m_- \\ &m_- = \frac{1-\Lambda}{\Lambda} m_c, \qquad \Lambda \text{ precipitation efficiency} \\ &m_c = \sigma_c \big((\mathrm{CAPE})^+ \big)^{1/2}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} m_e &= -(1-\sigma_c)(w_e)^- \\ (1-\sigma_c) w_e &= -(m_c + H_m u_x). \end{aligned} \tag{18}$$

In (16), (17), (18) above, the quantity $(X)^\pm$ denotes respectively the positive or negative part of the number X.

5. Coupling of the Stochastic CIN Model into the Parametrization

The equations in (14)–(18) are regarded here as the prototype deterministic GCM parametrization when discretized in a standard fashion utilizing central differences on a coarse mesh Δx with Δx ranging from 50 km to 250 km. In the simulations from (Khouider et al. 2003), $\Delta x = 80$ km. The coarse-grained stochastic CIN model is coupled to this basic parametrization. First, the area fraction for deep convection, σ_c, governing the upward mass flux strength, is allowed to vary on the coarse mesh and is given by

$$\sigma_c(j\Delta x) = \big[1 - \bar{\sigma}_I(j\Delta x) \big] \sigma_c^+ \tag{19}$$

with $\bar{\sigma}_I$ is the average in (2)

with σ_c^+ a threshold constant, $\sigma_c^+ = .002$ (Majda and Shefter 2001, Majda et al. 2004). When the order parameter σ_I signifies strong CIN locally so that $\bar{\sigma}_I = 1$, the flux of deep convection is diminished to zero while with PAC locally active, $\bar{\sigma}_I = 0$, this flux increases to the maximum allowed by the value σ_c^+. To complete the coupling of the stochastic CIN model into the parametrization, the coarse mesh external potential, $h_{\mathrm{ext}}(\vec{u}_j)$, from (11), (12), needs to be specified from the coarse mesh values, $\vec{u}_j$. There is no unique choice of the external potential but its form can be dictated by simple physical reasoning. In (Khouider et al 2003), the plausible physical assumption is made that when the convective downward mass flux, m_-, decreases, the energy for CIN decreases. Since the convective downward mass flux results from the evaporative cooling induced by precipitation falling into dry air, it constitutes a mechanisms which carries negatively buoyant cool and dry air from the middle troposphere onto the boundary layer hence tending to reduce CAPE and deep convection. Thus, the decreasing of this flux will allow the boundary layer to be able self-consistently reduce the convective inhibition so here

$$h_{\text{ext}}(j\Delta x, t) = m_{-}(j\Delta x, t). \tag{20}$$

The other parameters needed in the birth/death process are the characteristic time τ_I which varies over 5,10, and 20 days respectively in (Khouider et al 2003) while the microscale occupation fraction $l = 10$. This is consistent with small scale variation of CIN on the scale of eight kilometers. Finally, the strength of local interaction, βU_0, is systematically varied in (Khouider et al. 2003) from boundary layer interactions favoring CIN with $\beta U_0 > 0$ to those favoring PAC with $\beta U_0 < 0$. The parameter β is fixed to $\beta = 1$ so that variations in the mean interaction strength, U_0, will not directly alter the effects of the external field on the adsorption/desorption rates. This completes the specification of the coarse-grained stochastic model.

6. The Effects of Stochastic Parametrization on Climatology and Fluctuations

It is shown in (Khouider et al 2003) that the above stochastic parametrization of CIN can have a substantial effect on both the climatology and wave fluctuations in the idealized setting of a tropical Walker cell. See (Khouider et al 2003) for the details.

7. Advantages of This Coarse-Grained Stochastic Lattice Procedure

1) Retain systematically the energetics of unresolved features through the coarse-grained Gibb's measure
2) Has minimal computational overhead since there are rapid algorithms for updating birth death processes
3) Incorporates feedbacks of the resolved modes on the unresolved modes and there energetics through an external field
4) Includes dynamical coupling through not only sampling the probability distribution of unresolved variables but also their evolving behavior in time is constrained by the large scale dynamics

Besides shallow boundary layer clouds, the moist convective processes associated with three cloud types, congestus, deep convective, and stratiform, play a major role in interactions across scales in the tropics. Recently, the authors have developed deterministic model parametrizations with three cloud types with a number of attractive theoretical and observational features (Khouider and Majda 2005). The authors plan to extend the coarse-grained stochastic models to this context which is more realistic prototype for GCM'S in the near future.

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