The Lokal Modell (LM) of DWD / COSMO

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1 Introduction

The German Weather Service (Deutscher Wetterdienst, DWD) develops in common with his partners of the COSMO-group (**Co**nsortium for **S**mall-scale **Mo**delling) the mesoscale, non-hydrostatic limited-area atmospheric model 'Lokal Modell' (LM) to be used both for operational and research applications. COSMO was formed in October 1998 and consists now of the 5 countries Germany, Greece, Italy, Poland and Switzerland.

The lecture gave a short overview about the properties of LM and the planned new model LMK, this is described in the following section. Then a special topic in the newest LM development, the introduction of prognostic precipitation, will be presented. In the last section a somewhat more general topic is touched: the stability properties of different time-splitting schemes as the additive splitting and three versions of the Klemp-Wilhelmson splitting idea combined with the Euler-forward- and two Runge-Kutta-schemes.

2 The Lokal Modell (LM)

The dynamics of the LM bases on the non-hydrostatic, full compressible equations in advection form. A stationary, hydrostatic base state at rest is used to reduce the hydrostatic unbalance error in the calculation of the pressure gradient term. Therefore the prognostic variables are the three cartesian wind components u, v and w, pressure perturbation p' (deviation from the base state), temperature T, and five humidity variables, the specific masses for water vapor q^v, cloud water q^c, ice content q^i, rain q^r and snow q^s. Finally the turbulent kinetic energy (TKE) is used as a prognostic variable for the turbulence scheme. The coordinate system consists of rotated geographical coordinates with a generalized terrain-following height coordinate and a user defined vertical stretching. For the numerical treatment, the equations are discretized on an Arakawa-C grid in horizontal directions and a Lorenz grid in vertical direction. Therefore all the velocity components are staggered against the scalar variables. The discretization itself consists of second order finite differences. For the time-integration there are different choices: firstly (and currently the default option) the Klemp-Wilhelmson 3-timelevel Leapfrog scheme (Klemp and Wilhelmson, 1978) is implemented. Secondly, a 2-timelevel Runge-Kutta scheme of 2. order (Skamarock and Klemp, 1992) is implemented. Thirdly, a 3-timelevel 3-dimensional semi-implicit scheme can be chosen. Especially the Leapfrog scheme combined with the centered differences need some filtering: an Asselin filter to prevent from the checkerboard instability, a divergence filter (see below) to stabilize the whole time-splitting scheme, and a horizontal diffusion of 4. order to prevent from nonlinear instability and to smooth overshoots. In the current operational routine, LM uses horizontal grid sizes of about 7 km, a timestep of 40 seconds and $325 \times 325 \times 35$ gridpoints.

In the near future weather-predictions of the LM shall be accompanied by a very short range forecast system - the LMK-model ('LM-Kürzestfrist'). The goals of this development project is to deliver a model-based numerical weather prediction system to forecast severe weather events on the meso-γ-scale related to deep moist convection and interactions with fine-scale topography as super-, multicell thunderstorms, squall-lines or severe downslope winds, Föhnstorms and flash floodings. To this purpose an LM-version with a modified numerical core (two-timelevel-scheme) and improved physical parametrizations (6-class 'Graupel'-cloud physics scheme instead of the current 5-class-scheme, 3-dim. turbulence, ...) is applied on grid lengths of $\Delta x \approx 2.8$ km allowing a direct simulation of at least the large scale amounts of deep convection. 18-hour-forecasts shall be started every 3 hours from a rapid-update data assimilation cycle.

3 The new prognostic precipitation scheme

From LM version 3.9 (in operational use since 26.04.2004), the former diagnostic scheme for rain and snow was replaced by a so called prognostic precipitation scheme. The aim of this project was to improve the precipitation distribution in orographically structured areas due to horizontal drifting of rain and snow, solving the so-called 'windward-lee-problem'.

The conservation equation for humidity variables reads

$$\frac{\partial q^x}{\partial t} + \mathbf{v} \cdot \nabla q^x = -\nabla \cdot \mathbf{P}^x - \nabla \cdot \mathbf{F}^x + S^x \qquad (1)$$

where $q^x = \rho^x / \rho$ is the specific mass of the humidity variable x (ρ is the total density of air, $x = v, c, i, r, s$ as above). $\mathbf{P}^x$ are the sedimentation fluxes (of course only for $x = r$ and $x = s$). $\mathbf{F}^x$ are the turbulent fluxes (they are neglected up to now) and S^x denote all the transformation rates by cloud-microphysical processes.

In the former diagnostic scheme for LM, the left hand side of equation (1) for rain and snow was set to zero. This so called 'column-equilibrium-approximation' means that all rain and snow particles, which are generated by microphysical processes S^r and S^s fall down immediately (i.e. in the same timestep) vertically to the ground. Rain particles have an averaged fall velocity of about 5 m/s; if one asssumes, that they are generated for example at a height of 3 km, they need about 10 min. to fall down. In these 10 min. they are drifted horizontally by about 6 km (for an assumed wind speed of about 10 m/s). Normally wind velocities in greater heights are even higher and in the upper portions of the air, precipitation falls down as snow with a much lower sedimentation velocity of about 1..2 m/s, therefore, the drifting distances can reach several dozen kilometers. Therefore, the column-equlibrium approach is good for large-scale models, but no longer for mesoscale models as the LM. Improving the diagnostic scheme to a prognostic one simply means to implement the left hand side of equation (1) especially the advection terms for rain and snow.

The above mentioned 'windward-lee-problem' occurs in the vicinity of mountains and manifests in too strong precipitation maxima in the windward side, which also lie to far upstream and too few precipitation in the lee side. It was found, that the absent of the advection for the precipitationg particels was the main reason for theses problems. Smith (1979) distinguishes essentially three mechanisms for orographic precipitation generation: 1. the large-scale upslope precipitation. In this case an inflowing air mass is lifted by a mountain or mountaineous area, and clouds are generated above the condensation level. If the mountain is wide enough, then there is enough time to convert cloud droplets into rain particles and precipitation can occur. 2. In the seeder-feeder process again a cloud is generated by lifting, but if the mountain has not such a wide extent, there is no conversion into rain particles. Instead of there can be a washing out of the cloud particles by an overlying stratiform cloud layer which precipitates and therefore an intensification of the rainfall. 3. A triggering of cumulonimbus clouds can occur in unstable air masses by upslope winds over a mountain with heavy precipitation. In all of these different mechanisms there occurs also an overlaid drifting by the mean wind.

Advection is a rather time consuming numerical process, but there seems to be no abbreviation because of the very different mechanisms of orographic generating precipitation. We decided to implement a semi-Lagrange-scheme; the basic idea behind this huge class of advection schemes is the identity of (e.g. in one dimension)

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0 \quad \Leftrightarrow \quad \frac{d\phi}{dt} = 0 \text{ and } \frac{dx}{dt} = u. \tag{2}$$

This can be easily enhanced to three dimensions. The numerical formulation (for a three-timelevel scheme) is

$$\phi(x_j, t^{n+1}) = \phi(\tilde{x}_j^{n-1}, t^{n-1}), \tag{3}$$

where $\tilde{x}_j^{n-1}$ is the starting point of the trajectory at timelevel t^{n-1}, which ends up in x_j at timelevel t^{n+1}. Therefore the application of a semi-Lagrange scheme consists in two steps: 1. determine the backtrajectory, i.e. the point $\tilde{x}_j^{n-1}$. In principle, every solver for ordinary differential equation (systems) can be used to solve the equation $dx/dt = u$. Robert (1981) (e.g. in Staniforth and Côté, 1991) proposed to iterate the equation

$$a = \Delta t \; v(x_j - a, t^n) \tag{4}$$

with the displacement $2a = x_j - \tilde{x}_j^{n-1}$. This equation can be iterated; one can prove that one iteration step delivers a backtrajectory in 1. order Δt, and a second iteration step delivers one in 2. order Δt^2. The second step consists in interpolating the field ϕ at the starting point $\tilde{x}_j^{n-1}$ from the neighbouring grid points. The simplest interpolation would be a linear one. This is seldom used because of a strong numerical diffusion, but has the great advantage to be monotone and therefore positive definite, which is important for the transport of humidity variables. Quadratic interpolation was often used in the 60's and 70's. Nowadays cubic interpolation seems to be a good compromise between accuracy and numerical costs.

One of the most important reasons in using semi-Lagrange schemes is their stability: for constant velocity u and without source terms they are even unconditionally stable. For non-constant velocity fields u there is also no Courant stability criterion but stability is constraint by the velocity deformation. For the most velocity fields occuring in the atmosphere, this is not a serious constraint. In LM the original idea to use a semi-Lagrange-scheme in the transport of rain and snow was to handle the sedimentation term as an advection term too, which would result in vertical Courant numbers much bigger than one. The second reason is that this scheme shall also be used in the new model LMK, which also uses horizontal Courant numbers almost up to two. The first reason is now obsolete; the sedimentation term instead is discretized implicitly and together with the cloud microphysics scheme coupled by a simple Marchuk operator splitting with the Semi-Lagrange advection. We decided to use the Robert-iteration scheme up to the second iteration as is recommended in the literature (e.g. Staniforth and Côté, 1991). In idealized tests of a rotating plate velocity field we got much better conservation properties as with a backtrajectory of only first order. This backtrajectory calculation is rather time consuming in a staggered grid because of many averaging steps for the velocity components. The second step of interpolation was done with a simple trilinear interpolation scheme; this means a superposition of linear functions in each space direction which results in a polynom of the form

$$p(x,y,z) = c_1 + c_2 x + c_3 y + c_4 z + c_5 xy + c_6 xz + c_7 yz + c_8 xyz, \tag{5}$$

where the weights c_i are determined by the 8 neighbouring points. As mentioned the positive definiteness is an important property. The conservation properties are sufficient because rain and snow fall out in a relatively short time. For other particle types, like cloud water or cloud ice, trilinear interpolation would not be sufficient.

Because of the limited place here, we cannot present examples, but a case study can be found in the internet (Baldauf, 2004). The main result is, that the windward-lee-distribution could be improved in most cases. In winterly simulations of South-Germany, where mainly the Black forest as a midlevel mountaineous area induces errors in the windward-lee distribution, the spatial averaged precipitation was reduced by 15-25%, and the precipitation maxima were reduced by about 25-40%.

4 Time-splitting schemes in atmospheric models

In the numerical treatment of the atmosphere one is confronted with a wide range of time and velocity scales, starting with very slow processes due to the Coriolis force, whose zonal variation induces Rossby-waves, then gravity waves, then the normal advection transport up to the fast sound waves. In table 1 some of these processes are listed together with their characteristic velocity ranges and time scales.

process	velocity range	timestep (for $\Delta x \approx 3$ km)
Coriolis force (e.g. Rossby-waves)		$\tau_f \sim 50000$ s ($\sim 1/f$)
Advection	$c_a \sim 10 \dots 50$ m/s	$\tau_a \sim 60 \dots 300$ s
Buoyancy		$\tau_b \sim 600$ s ($\sim 2\pi/N$)
Gravity waves	$c_g \sim 10$ m/s	$\tau_g \sim 300$ s
Sound	$c_s \sim 330$ m/s	$\tau_a \sim 10$ s

Table 1: Some atmospheric processes with their characteristic velocity ranges and time scales. Where no natural time scale can be given, there is a time step calculated for a horizontal grid length of $\Delta x \approx 3$ km.

Nowadays there exist more or less well doing numerical schemes to treat the physical terms responsible for these single processes. The question now is how to combine these numerical schemes for processes with such very different time scales, especially if one is mainly interested in the 'slow' processes and not so much in the 'fast' ones (but which nevertheless have to be maintained by stability reasons)? We restrict ourselves to two such processes (which themself can consist of several processes with more or less equal time scales)

$$\frac{\partial}{\partial t} q = \mathscr{P}_s q + \mathscr{P}_f q, \tag{6}$$

where q stands for a vector of fields, for example $q = (u, w, T', p')$ for the sound-buoyancy-advection-system below, $\mathscr{P}_s$ denotes the (differential) operator of the slow processes (like advection, diffusion, Coriolis force) and $\mathscr{P}_f$ denotes the operator of the fast processes (at least sound processes).

There exist of course several integration methods. First one could integrate all single processes with such a small timestep that they all can be integrated in a stable manner. If one combines this with a simple operator splitting (i.e. calculate one process, update the fields, then calculate the next process with these updated fields, and so on) this in most cases gives a stable scheme (but not necessarily). This procedure is in most cases hopelessly inefficient. A second method would be to find unconditionally stable discretizations for the fast processes - normally purely implicit schemes. Again one can combine them by operator splitting and can choose a timestep dictated only by the slow processes. This method becomes more and more popular due to increasing computer capacities. A third way, which shall be discussed in the following, is to use time-splitting methods. They calculate the slow processes with a big timestep ΔT and the fast ones with a small timestep Δt. This is much more efficient as the above proposed first way, because the fast processes (in atmospheric models usually the sound and perhaps the buoyancy terms) can be calculated computationally cheap.

Before we discuss some time-splitting schemes, we introduce some denotations. We introduce numerical operators Q and P in such a way that one timestep (for two-timelevel-schemes) from t to $t + \Delta T$ for the slow processes is denoted as

$$q^{t+\Delta T} = Q_{s,\Delta T}(q^t) \quad = q^t + \Delta T \, P_s(q^t) + \Delta T^2 P_s^{(2)}(q^t) + \dots \tag{7}$$

and one timestep from t to $t + \Delta t$ as

$$q^{t+\Delta t} = Q_{f,\Delta t}(q^t) \quad = q^t + \Delta t \;\, P_f(q^t) + \Delta t^2 \;\, P_f^{(2)}(q^t) + \dots \tag{8}$$

The time-splitting ratio is defined as $n_s = \Delta T / \Delta t$.

For any numerical scheme there are to be fulfilled at least two conditions. The scheme has to be consistent, that means in the limit of vanishing timesteps and grid lengths the difference equations reduce to the original differential equations. In our operator denotation this means that the operator of the whole scheme reads

$$Q_{tot} = 1 + \Delta T\, P_s + n_s \Delta t\, P_f + O(\Delta T^2, \Delta t^2). \tag{9}$$

The second condition is stability, which can be analysed by von-Neumann stability analysis (e.g. Durran, 1998). For this, one has to linearize the PDE-System for $u(x,z,t)$, $w(x,z,t)$, ... (e.g. in two dimensions) around a base state and tries to choose the coefficients as constant. In its discretization for the fields u^n_{jl}, w^n_{jl}, ... at grid points (x_j, z_l) and at timelevel t^n with grid sizes Δx, Δz one inserts the single Fourier-modes e.g.

$$u^n_{jl} = u^n\, e^{i(k_x j\Delta x + k_z l \Delta z)}. \tag{10}$$

For two-timelevel-schemes one then gets an expression like $q^{n+1} = A q^n$. The stability range is determined by the eigenvalues λ_i of the amplification matrix A: numerical stability is given if all absolute eigenvalue $|\lambda_i| < 1$. Of course there are a lot of other requirements for numerical schemes like accuracy, form fidelity (especially for advection schemes), conservation properties or efficiency. But a minium requirement for a numerical scheme is convergence, and this is stated by the Lax equivalence theorem (at least for linear PDE-systems) if it is consistent and stable.

Now we want to discuss some time-splitting schemes. A simple method is the *additive splitting*, sometimes also called complete operator splitting, method of fractional steps or even multiplicative splitting. The operator splitting idea was perhaps first introduced by Peaceman and Rachford (1955) in their alternating direction implicit (ADI)-method and later on introduced in the meterological modelling by Marchuk (1974). Transformed to the time-splitting idea it means that one calculates first the slow processes in a big timestep, updates the fields, and then carries out n_s small timesteps with the fast processes, again with successive updating (figure 1). In our operator formulation it means

$$q^* = Q_s(q^t) \tag{11}$$

$$q^{t+\Delta T} = Q_f(Q_f(\ldots Q_f(q^*)\ldots)) \tag{12}$$

The numerical costs are obviously one times calculation of Q_s and n_s times calculation of Q_f. Leveque and Oliger (1983) proved that stability is guarantied only, if P_s and P_f both are stable *and* commutable; the last requirement is normally not fulfilled in atmospheric models. The scheme is only of first order in ΔT (for non-commutable operators). This can be cured by the method of Strang

$$q^{t+\Delta T} = (Q_f)^{n_s/2} \cdot Q_s \cdot (Q_f)^{n_s/2}\, q^t \tag{13}$$

to produce a scheme of 2. order in ΔT. Purser and Leslie (1991) (also in Durran, 1998) found that additive splitting is rather noisy, which has its cause in a relatively weak coupling between the slow and fast modes and is perhaps the main reason why it is not widely applied in meteorological models.

This coupling problem was circumvented by the basic idea of Klemp and Wilhelmson (1978). Again, in a first step the slow processes are calculated. But now the tendency of these processes is stored and added in each small timestep (figure 1). If one combines this *Klemp-Wilhelmson-(KW)-time-splitting* idea (sometimes called partial operator splitting) with a simple Euler-forward-(EF)-method then we get in operator notation for the big timestep

$$q^* = Q_s(q^t), \quad dq_s = \frac{q^* - q^t}{\Delta T} \tag{14}$$

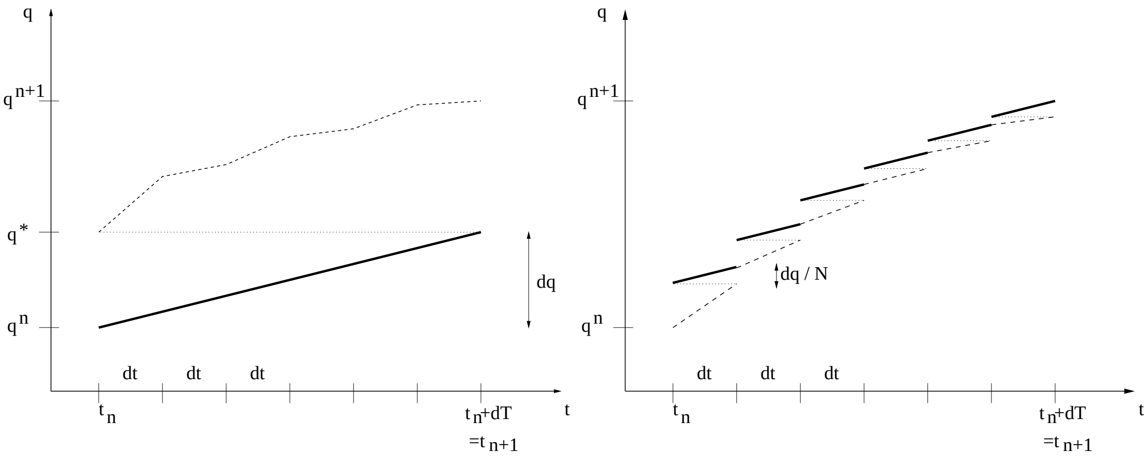

Figure 1: Schematic application of the additive time-splitting (left) and the Klemp-Wilhelmson-Euler-forward-time-splitting (right)

and for the small timesteps

$$q^{t+\Delta t} = Q_f(q^t) + \Delta t\, dq_s \tag{15}$$
$$q^{t+2\Delta t} = Q_f(q^{t+\Delta t}) + \Delta t\, dq_s \tag{16}$$
$$\dots$$
$$q^{t+\Delta T} = Q_f(q^{t+(n_s-1)\Delta t}) + \Delta t\, dq_s \tag{17}$$

Again this is a two timelevel scheme and the numerical costs are in principle the same as for the additive time-splitting. But Skamarock and Klemp (1992) claimed that this scheme would be not stable. That this is not entirely true can be shown by a stability analysis of the linearised, one-dimensional sound-advection-equations (see e.g. Tatsumi (1983), Skamarock and Klemp (1992) or Wicker and Skamarock (1998))

$$\frac{\partial u}{\partial t} + c_A \frac{\partial u}{\partial x} = -c_S \frac{\partial p}{\partial x} \tag{18}$$
$$\frac{\partial p}{\partial t} + c_A \frac{\partial p}{\partial x} = -c_S \frac{\partial u}{\partial x} \tag{19}$$

It consists of only two single processes, a slow advection and a fast sound propagation, and has the special feature that both operators commute. Therefore an additive time-splitting is stable. For a stability analysis one has to specify the discretizations of the single processes. For the advection we use a simple upwind scheme of 1. order which is stable for Courant numbers $C_A < 1$, where $C_A = c_A\, \Delta T / \Delta x$. The sound processes are discretized by a forward-backward scheme as described in Mesinger (1977), which has a stable range for sound-Courant-numbers $C_S < 2$ on an unstaggered grid and $C_S < 1$ on a staggered grid ($C_S = c_S \Delta t / \Delta x$).

If one calculates the stability range (by the above mentioned von-Neumann-analysis) for the KW-EF-scheme one finds that the $2\Delta x$ waves are stable for $C_A < 1$ and the long waves ($k \to 0$) are stable for $C_A < 1 - C_S(1 + n_s)$ (Baldauf, 2002). There is a stable range but it is very small and completely limited by the long waves; therefore a smoothing filter would not cure the problem. But one could try a divergence filter as proposed by Skamarock and Klemp (1992).

A divergence filter is simply a gradient of the velocity divergence added on the right side of the velocity equation. In vector notation this means

$$\frac{\partial \mathbf{v}}{\partial t} + \dots = \dots + \alpha_{div} \nabla (\operatorname{div} \mathbf{v}). \tag{20}$$

If one takes the divergence of the velocity equation, some sort of diffusion equation for the divergence arises. With the diffusion coefficient α_{div} one can define a 'divergence-Courant-number' $C_{div,x} = \alpha_{div}\Delta t/\Delta x^2$. If the divergence filter is explicitly discretized, it is only stable for $C_{div,x} < 1/2$ (in one dimension). Application of a divergence filter is seen as not so critical because in mesoscale models the only waves which are connected with divergence are the sound waves.

If one introduces this divergence damping into the KW-EF-scheme the stability of the long waves ($k \to 0$) has increased to (Baldauf, 2002)

$$1 - C_A - C_S > \left(C_S - \frac{C_{div}}{C_A} \right) n_s \tag{21}$$

and in the case of strong divergence damping long waves are stable for (see figure 2)

$$C_{div} > C_S C_A \qquad \text{and} \qquad n_s > \frac{1 - C_A - C_S}{C_S - \frac{C_{div}}{C_A}}. \tag{22}$$

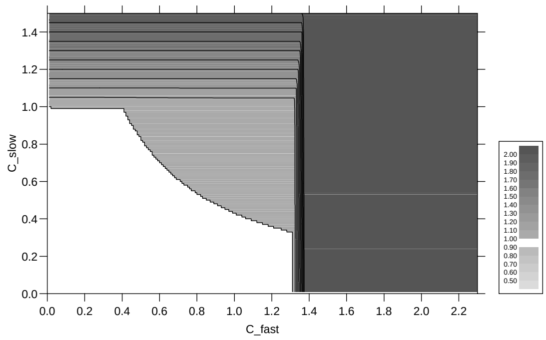

Figure 2: Maximum eigenvalue $|\lambda|_{max}(C_S, C_A)$ of the one-dimensional sound-advection system for the Klemp-Wilhelmson-Euler-Forward-scheme with divergence damping, $C_{div} = 0.45$, $n_s = 8$.

Therefore we can conclude that the KW-EF-scheme can be stabilized by using a strong divergence damping. The reason why Skamarock and Klemp (1992) found an instability was, that they carried out a spatial Fourier transform for their stability analysis, which in this connection seems to be too careful. As can be seen here, only a complete analysis which considers also the spatial discretization delivers the stability range. It is interesting that there is no stability constraint for the time-splitting ratio in the 1-dim. sound-advection-system.

The original method of Klemp and Wilhelmson (1978) did not use the EF-scheme but the Leapfrog method. The Leapfrog-scheme for an ODE $dq/dt = f(q)$ reads

$$q^{n+1} = q^{n-1} + 2\Delta T \; f(q^n). \tag{23}$$

This combined with the time-splitting idea results again in two steps analoguous to the KW-EF-scheme: in a first step calculate the tendency of slow processes with q^t and in a second step perform a KW-EF-splitting but now from $t - \Delta T$ to $t + \Delta T$ with this slow tendency. In contrast to the former time-splitting schemes this *KW-Leapfrog-scheme* is a 3-timelevel method. Because of the centered differencing in time of the Leapfrog-method there can be used only special advection schemes, also centered differences. The numerical costs of this scheme are higher because the fast processes have to be calculated twice (1 times Q_s and $2n_s$ times Q_f). But the great advantage is that Skamarock and Klemp (1992) could show, that this scheme is stable with an Asselin-filter and with divergence damping.

The drawbacks of the 3-timelevel scheme with its limitations in using advection schemes led Wicker and Skamarock (1998) to use a Runge-Kutta-scheme of 2. order (RK2) for the time integration instead of EF or

Leapfrog. An RK2-scheme for an ODE $dq/dt = f(q)$ reads

$$q^* = q^n + \frac{\Delta T}{2} f(q^n) \tag{24}$$

$$q^{n+1} = q^n + \Delta T \; f(q^*) \tag{25}$$

and a *KW-RK2-method* can be built up with the time-splitting idea by combining two KW-EF-steps: at first a KW-EF-step from t to $t + \Delta T/2$ which gives q^* as a new state, then calculate the slow tendency with q^* and perform another KW-EF-step from t to $t + \Delta T/2$ with this slow tendency. This method has the advantage of being a 2-timelevel-scheme, but its numerical costs are again higher: for one timestep one needs 2 times Q_s and $1.5 n_s$ times P_f. To circumvent these higher costs Gassmann (2002) proposed a *'shortened' KW-RK2-scheme* by doing the first KW-EF-step (the first RK-step) only with the fast processes.

Of course one can play this game further on: Wicker and Skamarock (2002) proposed a *KW-RK3-splitting* based on a 3-step RK-scheme; which interestingly converges rather fast, despite the fact that it is formally not of 3. order (compare e.g. Hundsdorfer et al. (1995) for some formal aspects of RK-schemes). The main advantage is, that with this scheme one can reach much higher Courant-numbers by using an upwind 5. order advection method ($C_A \approx 1.42$ compared to $C_A \approx 0.3$ with KW-RK2). Of course this scheme is more costly: it uses 3 times Q_s and $1.833 n_s$ times Q_f per timestep.

A question we want to address now is: can time-splitting schemes deliver a correct stationary solution? To get a first insight we discuss a simple example: the scalar relaxation equation with an external force (Murthy and Nanjundiah, 2000)

$$\frac{d\phi}{dt} = \underbrace{-\beta\phi}_{Relax.} + \underbrace{g}_{Force}, \quad \beta > 0, \; \beta, g \text{ constant}. \tag{26}$$

For constant coefficients this ODE has obviously the stationary solution $\phi_s = g/\beta$. For this equation one can analytically calculate the numerical stationary solution of the different time-splitting schemes (assuming that the single processes are discretized by Euler-forward-schemes). The result is shown in table 2, where in the one case the relaxation term is seen as the fast process, in the other case as the slow process. Of course all the schemes deliver consistent solutions for $\Delta t, \; \Delta T \to 0$, but in practice one wants to have an as big as much timestep: $\beta \Delta t \approx 1$ or $\beta \Delta T \approx 1$. Obviously the additive splitting and also the KW-'shortened'-RK2-scheme then do not deliver the correct stationary solution.

Splitting	Relax. fast, Forcing slow	Forcing fast, Relax. slow
Additive	$\frac{g}{\beta}\left(1 + \frac{1}{2}\frac{n_s - 1}{n_s}\beta\,\Delta t + \ldots\right)$	$\frac{g}{\beta}(1 - \beta\,\Delta T)$
KW-EV	$\frac{g}{\beta}$	$\frac{g}{\beta}$
KW-Leapfrog	$\frac{g}{\beta}$	$\frac{g}{\beta}$
KW-RK2	$\frac{g}{\beta}$	$\frac{g}{\beta}$
KW-RK2-short	$\frac{g}{\beta}$	$\frac{g}{\beta}(1 - \beta\,\frac{\Delta T}{2})$

Table 2: Numerical stationary solution for the different time-splitting schemes for the scalar relaxation equation

In most papers the above mentioned sound-advection system is inspected with the argument that the interaction of these two types of hyperbolic waves is the most severe constraint in the stability of time-splitting schemes. In the atmospheric modelling there are also buoyancy forces and it is not clear if the gravity waves are fast or slow processes. Therefore we want to inspect a linearised, 2-dimensional sound-buoyancy-advection-system:

$$
\begin{array}{lclllr}
\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} & = & -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} & & +Q_x & (27) \\
\frac{\partial w}{\partial t} + U_0 \frac{\partial w}{\partial x} & = & -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} & -\left(\frac{p'}{p_0} - \frac{T'}{T_0}\right) g & +Q_z & (28) \\
\frac{\partial p'}{\partial t} + U_0 \frac{\partial p'}{\partial x} & = & -\frac{c_p}{c_v} p_0 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) & +\rho_0 g w & & (29) \\
\frac{\partial T'}{\partial t} + U_0 \frac{\partial T'}{\partial x} & = & -\frac{R}{c_v} T_0 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) & -\frac{\partial T_0}{\partial z} w & & (30) \\
\text{Tendency Adv.} & & \text{Sound} & \text{Buoyancy} & \text{Filtering} &
\end{array}
$$

These equations are formulated in the primitive variables u, w, $p = p_0 + p'$ and $T = T_0 + T'$. Q_x and Q_z are the divergence damping terms

$$
Q_x = \alpha_{div} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right), \quad Q_z = \alpha_{div} \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right). \tag{31}
$$

For the following stability analysis the buoyancy terms are discretized by a forward-backward scheme. This is stable for a 'buoyancy'-Courant-number $C_{buoy} = \omega_a \Delta t < 2$, where the acoustic cut-off frequency $\omega_a = \sqrt{N^2 + g^2/c_S^2} \approx 0.03$ 1/s is introduced. (an additional dimensionless parameter is $C_\beta = c_S^2/g\, \partial \log T_0/\partial z$).

In figures 3-6 stability diagrams are presented for this system. Again the maximum absolute value of the eigenvalues in dependency of the advection Courant-number C_A (with the base state velocity in horizontal direction) and the sound Courant-number C_S is shown. In all the cases a divergence Courant number $C_{div,x} = C_{div,z} = 0.23$ was chosen, the time splitting ratio $n_s = 8$ and the stratification was that of a standard atmosphere, this results in the dimensionless parameter $C_\beta = -0.24$.

We start with an isotropic grid $\Delta x = \Delta z$, therefore the horizontal and vertical sound Courant numbers are equal, $C_{S,x} = C_{S,z}$. In figures 3 the buoyancy terms are calculated in the fast processes, therefore the buoyancy Courant number is calculated with the small time step. $C_{buoy} = 0.01$ was chosen, which results in a small timestep of about 0.3 sec. Therefore this is the case of a very small scale atmospheric model. The KW-EF-scheme has a reduced stability range compared to KW-RK2 and KW-shortened-RK2 by a slight instability for small C_S. The strong reduction of stability $C_S < 0.2$ is due to the staggered grid; compare this with the unstaggered case in the one-dimensional sound-advection-system (figure 2) where $C_S < 1.3$ is possible. This reduction in the staggered case is due to the $2\Delta x$-waves and can be cured to some extend by a smoothing filter.

In figures 4 the $C_{buoy} = 0.1$ was chosen. This results in a small time step of $\Delta t \approx 3$s which is in the range of near future weather prediction models. Obviously, the KW-EF-scheme breaks down and also the KW-shortened-RK2-scheme has a reduced stability range. The KW-RK2-scheme has no reduction of stability; it seems to be rather independent from C_{buoy}.

Figures 5 show the case that the buoyancy processes now are calculated in the slow processes. Therefore we have to calculate the buoyancy-Courant number with the big timestep and the case $C_{buoy} = 0.1$ has to be compared rather with the case in 3. Obviously the KW-shortened-RK2 now suffers from a slight instability, this could be a hint to handle buoyancy as a fast process rather than a slow process, in spite of the fact that its time scale $1/N$ is rather big.

The last figures 6 shows the more realistic case of an anisotropic grid with $\Delta x = 10\, \Delta z$. Here the sound processes are handled vertically implicit as is done in the most atmospheric models with time-splitting. This also increases the possible sound-Courant number $C_S \equiv C_{S,x}$ because now the sound stability condition is again a one-dimensional restriction and an arbitrary $C_{S,z}$ is possible. Obviously, the KW-EF-scheme has a strongly

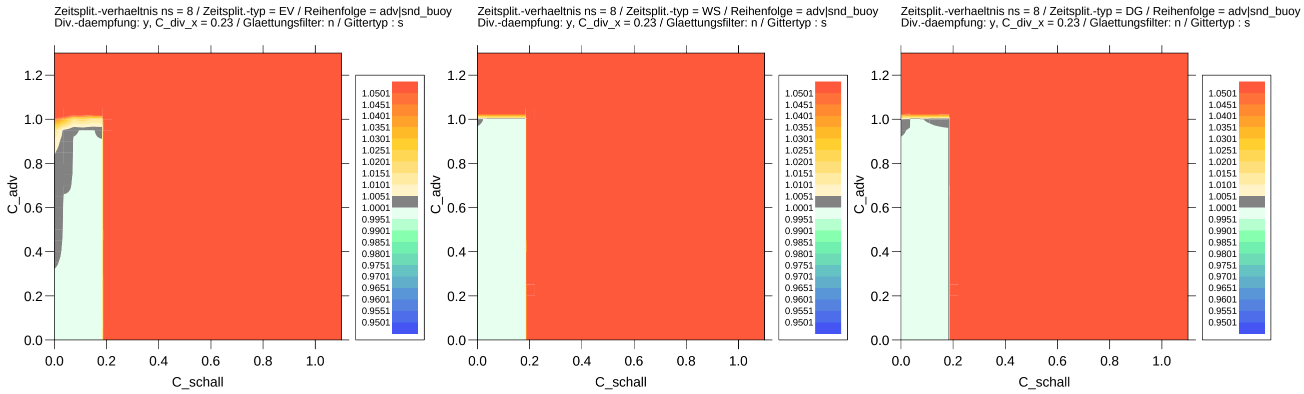

Figure 3: Stability diagram i.e. the maximum absolute eigenvalue of the sound-advection-buoyancy-system in dependency of the Courant numbers for advection C_A and sound C_S. KW-EF-scheme (left), KW-RK2-scheme (middle), KW-shortened-RK2 (right). $C_{buoy} = 0.01$, divergence damping $C_{div,i} = 0.23$, $n_s = 8$.

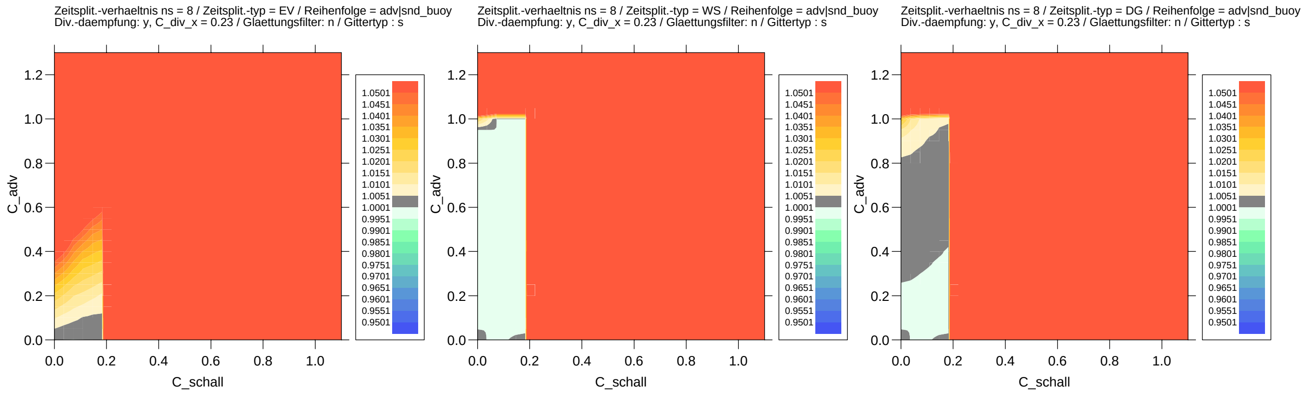

Figure 4: As in figure 3, but for $C_{buoy} = 0.1$

reduced stability range. The KW-RK2-scheme has the largest stability range, but as mentioned compared to KW-shortened-RK2 one has a double calculation of the slow processes.

To conclude, the KW-EF needs the lowest number of operations per timestep but has only a limited stability range. The KW-shortened-RK2-scheme is relatively stable but in practice has no good convergence properties as shown by the scalar relaxation equation. The best stability and convergence properties has the KW-RK2-scheme.

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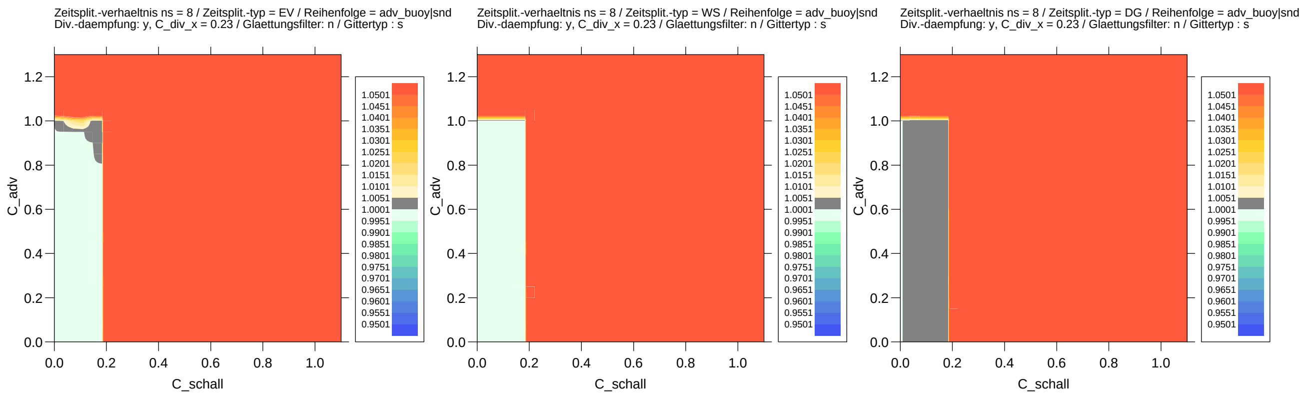

Figure 5: As in figure 3, but for $C_{buoy} = 0.1$, buoyancy in slow processes

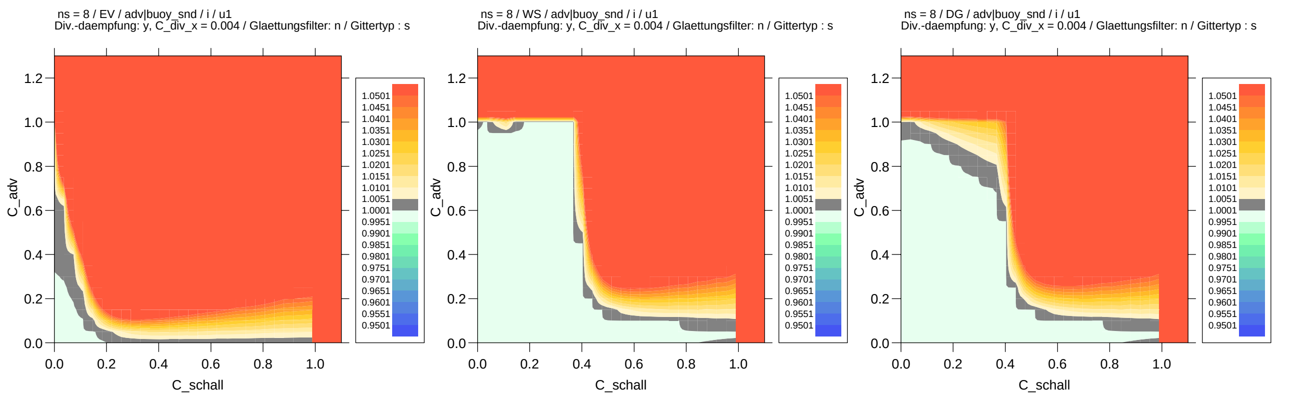

Figure 6: As in figure 3, but sound scheme vertically implicit, and anisotropic grid $\Delta x = 10\,\Delta z$

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