Subgrid-scale orographic drag

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1. General principles

The influence of subgridscale orography on the momentum of the atmosphere, and hence on other parts of the physics, is represented by a combination of lower-troposphere drag created by orography assumed to intersect model levels, and vertical profiles of drag due to the absorbtion and/or reflection of vertically propagating gravity waves generated by stably stratified flow over the subgridscale orography. The scheme is described in detail in *Lott and Miller* (1996).

The scheme is based on ideas presented by *Baines and Palmer* (1990), combined with ideas from bluff-body dynamics. The assumption is that the mesoscale-flow dynamics can be described by two conceptual models, whose relevance depends on the non-dimensional height of the mountain via.

$$H_{\mathrm{n}} = \frac{NH}{|U|} \tag{1}$$

where H is the maximum height of the obstacle, U is the wind speed and N is the Brunt–Väisälä frequency of the incident flow.

At small H_{n} all the flow goes over the mountain and gravity waves are forced by the vertical motion of the fluid. Suppose that the mountain has an elliptical shape and a height variation determined by a parameter b in the along-ridge direction and by a parameter a in the cross-ridge direction, such that $\gamma = a/b \le 1$, then the geometry of the mountain can be written in the form

$$h(x, y) = \frac{H}{1 + x^2/a^2 + y^2/b^2} . \tag{2}$$

In the simple case when the incident flow is at right angles to the ridge the surface stress due to the gravity wave has the magnitude

$$\tau_{\text{wave}} = \rho_0 bGB(\gamma)NUH^2 \tag{3}$$

provided that the Boussinesq and hydrostatic approximations apply. In Eq. (3) G is a function of the mountain sharpness (*Phillips* 1984), and for the mountain given by Eq. (2), $G \approx 1.23$. The term $B(\gamma)$ is a function of the mountain anisotropy, γ, and can vary from $B(0) = 1$ for a two-dimensional ridge to $B(1) = \pi/4$ for a circular mountain.

At large H_{n}, the vertical motion of the fluid is limited and part of the low-level flow goes around the mountain. As is explained in Section 2, the depth, Z_{blk}, of this blocked layer, when U and N are independent of height, can be expressed as

$$Z_{\text{blk}} = H \times \max\left(0, \frac{H_{\text{n}} - H_{\text{n}_{\text{crit}}}}{H_{\text{n}}}\right) \tag{4}$$

where $H_{\text{n}_{\text{crit}}}$ is a critical non-dimensional mountain height of order unity. The depth Z_{blk} can be viewed as the upstream elevation of the isentropic surface that is raised exactly to the mountain top. In each layer below Z_{blk} the flow streamlines divide around the obstacle, and it is supposed that flow separation occurs on the obstacle's flanks. Then, the drag, $D_{\text{blk}}(z)$, exerted by the obstacle on the flow at these levels can be written as

$$D_{\text{blk}}(z) = -\rho_0 C_{\text{d}} l(z) \frac{U|U|}{2} \tag{5}$$

Here $l(z)$ represents the horizontal width of the obstacle as seen by the flow at an upstream height z and C_{d}, according to the free streamline theory of jets in ideal fluids, is a constant having a value close to unity (*Kirchoff* 1876; *Gurevitch* 1965). According to observations, C_{d} can be nearer 2 in value when suction effects occur in the rear of the obstacle (Batchelor 1967). In the proposed parametrization scheme this drag is applied to the flow, level by level, and will be referred to as the drag of the 'blocked' flow, D_{blk}. Unlike the gravity-wave-drag scheme, the total stress exerted by the mountain on the 'blocked' flow does not need to be known *a priori*. For an elliptical mountain, the width of the obstacle, as seen by the flow at a given altitude $z < Z_{\text{blk}}$, is given by

$$l(z) = 2b\left(\frac{Z_{\text{blk}} - z}{z}\right)^{1/2} \tag{6}$$

In Eq. (6), it is assumed that the level Z_{blk} is raised up to the mountain top, with each layer below Z_{blk} raised by a factor H/Z_{blk}. This leads, effectively, to a reduction of the obstacle width, as seen by the flow when compared with the case in which the flow does not experience vertical motion as it approaches the mountain. Then applying Eq. (5) to the fluid layers below Z_{blk}, the stress due to the blocked-flow drag is obtained by integrating from $z = 0$ to $z = Z_{\text{blk}}$, viz.

$$\tau_{\text{blk}} \approx C_{\text{d}} \pi b \rho_0 Z_{\text{blk}} \frac{U|U|}{2}. \tag{7}$$

However, when the non-dimensional height is close to unity, the presence of a wake is generally associated with upstream blocking and with a downstream foehn. This means that the isentropic surfaces are raised on the windward side and become close to the ground on the leeward side. It we assume that the lowest isentropic surface passing over the mountain can be viewed as a lower rigid boundary for the flow passing *over* the mountain, then the distortion of this surface will be seen as a source of gravity waves and, since this distortion is of the same order of magnitude as the mountain height, it is reasonable to suppose that the wave stress will be given by Eq. (3), whatever the depth of the blocked flow, Z_{blk}, although it is clearly an upper limit to use the total height, H. Then, the total stress is the sum of a wave stress, τ_{wave}, and a blocked-flow stress whenever the non-dimensional mountain height $H_{\mathrm{n}} > H_{\mathrm{n_{crit}}}$, i.e.

$$\tau \approx \tau_{\mathrm{wave}}\left\{1 + \frac{\pi C_{\mathrm{d}}}{2GB(\gamma)}\max\left(0, \frac{H_{\mathrm{n}} - H_{\mathrm{n_{crit}}}}{H_{\mathrm{n}}^2}\right)\right\} . \tag{8}$$

The addition of low-level drag below the depth of the blocked flow, Z_{blk}, enhances the gravity-wave stress term in Eq. (8) substantially.

In the present scheme the value of C_{d} is allowed to vary with the aspect ratio of the obstacle, as in the case of separated flows around immersed bodies (*Landweber* 1961), while at the same time setting the critical number $H_{\mathrm{n_{crit}}}$ equal to 0.5 as a constant intermediate value. Note also that for large H_{n}, Eq. (8) overestimates the drag in the three-dimensional case, because the flow dynamics become more an more horizontal, and the incidence of gravity waves is diminished accordingly. In the scheme a reduction of this kind in the mountain-wave stress could have been introduced by replacing the mountain height given in Eq. (3) with a lower 'cut-off' mountain height, $H(H_{\mathrm{n_{crit}}}/H_{\mathrm{n}})$. Nevertheless, this has not been done. Cases with large non-dimensional mountain heights are often associated with low-level wave breaking, and hence the main impact of adopting of a cut-off mountain height would be a reduction of this low-level drag.

2. Description of the scheme

Following *Baines and Palmer* (1990), the subgrid-scale orography over one grid-point region is represented by four parameters μ, γ, σ and θ which stand for the standard deviation, the anisotropy, the slope and the geographical orientation of the orography, respectively. These four parameters have been calculated from the US Navy (USN) ($10' \times 10'$) data-set.

The scheme uses values of low-level wind velocity and static stability which are partitioned into two parts. The first part corresponds to the incident flow which passes over the mountain top, and is evaluated by averaging the wind, the Brunt–Väisälä frequency and the fluid density between μ and 2μ above the model mean orography. Following *Wallace et al.* (1983), 2μ is interpreted as the envelope of the subgrid-scale mountain peaks above the model orography. The wind, the Brunt–Väisälä frequency and the density of this part of the low-level flow will be labelled U_H, N_H and ρ_H, respectively. The second part is the 'blocked' flow, and its evaluation is based on a very simple interpretation of the non-dimensional mountain height H_{n}. To first order in the mountain amplitude, the obstacle excites a wave, and the sign of the vertical displacement of a fluid parcel is controlled by the wave phase. If a fluid parcel ascends the upstream mountain flank over a height large enough to significantly modify the wave phase, its vertical displacement can become zero, and it will not cross the mountain summit. In this case the blocking height, Z_{blk}, is the highest level located below the mountain top for which the phase change between Z_{blk} and the mountain top exceeds a critical value $H_{\mathrm{n_{crit}}}$, i.e.

$$\int_{Z_{\mathrm{bl;k}}}^{3\mu} \frac{N}{U_{\mathrm{p}}}\,\mathrm{d}z \geq H_{\mathrm{n_{crit}}} \tag{9}$$

In the inequality (9), the wind speed, $U_{\mathrm{p}}(z)$, is calculated by resolving the wind, $U(z)$, in the direction of the flow U_H. Then, if the flow veers or backs with height, (9) will be satisfied when the flow becomes normal to U_H. Levels below this 'critical' altitude define the low-level blocked flow. The inequality (9) will also be satisfied below inversion layers, where the parameter N is very large. These two properties allow the new parametrization scheme to mimic the vortex shedding observed when pronounced inversions occur (*Etling* 1989). The upper limit in the equality (9) was chosen to be 3μ, which is above the subgrid-scale mountain tops. This ensures that the integration in equality (9) does not lead to an underestimation of Z_{blk}, which can occur because of the limited vertical resolution when using 2μ as an upper limit (a better representation of the peak height), but this upper limit could be relaxed given better vertical resolution.

In the following subsection the drag amplitudes will be estimated combining formulae valid for elliptical mountains with real orographic data. Considerable simplifications are implied and the calculations are, virtually, scale analyses relating the various amplitudes to the sub-grid parameters.

2.1 Blocked-flow drag

Within a given layer located below the blocking level Z_{blk}, the drag is given by Eq. (5). At a given altitude z, the intersection between the mountain and the layer approximates to an ellipse of eccentricity

$$(a', b') \approx (a, b)\left(\frac{Z_{\text{blk}} - z}{z + \mu}\right)^{\frac{1}{2}}, \tag{10}$$

where, by comparison with Eq. (6), it is also supposed that the level $z = 0$ (i.e. the model mean orography) is at an altitude μ above the mountain valleys. If the flow direction is taken into account, the length $l(z)$ can be written approximately as

$$l(z) \approx 2 \max\,(b\cos\psi, a\sin\psi)\left(\frac{Z_{\text{blk}} - z}{z + \mu}\right)^{\frac{1}{2}} \tag{11}$$

where ψ is the angle between the incident flow direction and the normal ridge direction, θ, For one grid-point region and for uniformly distributed subgrid-scale orography, the incident flow encounters $L/(2a)$ obstacles is normal to the ridge $(\psi = 0)$, whereas if it is parallel to the ridge $(\psi = \pi/2)$ it encounters $L/(2b)$ obstacles, where L is the length scale of the grid-point region. If we sum up these contributions, the dependence of Eq. (11) on a and b can be neglected, and the length $l(z)$ becomes

$$l(z) = L\left(\frac{Z_{\text{blk}} - z}{z + \mu}\right)^{\frac{1}{2}}. \tag{12}$$

Furthermore, the number of consecutive ridges (i.e. located one after the other in the direction of the flow) depends on the obstacle shape: there are approximately $L/(2b)$ successive obstacles when the flow is along the ridge, and $L/(2a)$ when it is normal to the ridge. If we take this into account, together with the flow direction, then

$$l(z) = \frac{L^2}{2}\left(\frac{Z_{\text{blk}} - z}{z + \mu}\right)^{\frac{1}{2}} \max\left(\frac{\cos\psi}{a}, \frac{\sin\psi}{b}\right). \tag{13}$$

Relating the parameters a and b to the subgrid-scale orography parameters $a \approx \mu/\sigma$ and $a/b \approx \gamma$ and, allowing the drag coefficient to vary with the aspect ratio of the obstacle as seen by the incident flow, we have

$$r = \frac{\cos^2\psi + \gamma\sin^2\psi}{\gamma\cos^2\psi + \sin^2\psi} , \tag{14}$$

and the drag per unit area and per unit height can be written

$$D_{\mathrm{blk}}(z) = -C_{\mathrm{d}} \max\left(2 - \frac{1}{r} , 0\right) \rho\frac{\sigma}{2\mu} \left(\frac{Z_{\mathrm{blk}} - z}{z + \mu}\right)^{\frac{1}{2}} \max(\cos\psi , \gamma\sin\psi)\frac{U|U|}{2} . \tag{15}$$

The drag coefficient is modulated by the aspect ratio of the obstacle to account for the fact that C_{d} is twice as large for flow normal to an elongated obstacle as it is for flow round an isotropic obstacle. The drag tends to zero when the flow is nearly along a long ridge because flow separation is not expected to occur for a configuration of that kind. It can be shown that the term $\max(\cos\psi , \gamma \sin\psi)$ is similar to a later form used for the directional dependence of the gravity-wave stress. For simplicity, this later form has been adopted, i.e.

$$D_{\mathrm{blk}}(z) = C_{\mathrm{d}} \max\left(2 - \frac{1}{r},0\right) \rho\frac{\sigma}{2\mu} \left(\frac{Z_{\mathrm{blk}} - z}{z + \mu}\right)^{\frac{1}{2}} (B\cos^2\psi + C\sin^2\psi)\frac{U|U|}{2} \tag{16}$$

where the constants $B(\gamma)$ and $C(\gamma)$ are defined below. The difference between Eq. (15) and Eq. (16) has been shown to have only a negligible impact on all aspects of the model's behaviour,

In practice, Eq. (16) is suitably resolved and applied to the component from of the horizontal momentum equations. This equation is applied level by level below Z_{blk} and, to ensure numerical stability, a quasi-implicit treatment is adopted whereby the wind velocity U in Eq. (16) is evaluated at the updated time $t + \mathrm{d}t$, while the wind amplitude, $|U|$, is evaluated at the previous time step.

2.2 Gravity-wave drag

This gravity-wave part of the scheme is based on the work of *Miller et al.* (1989) and *Baines and Palmer* (1990), and takes into account some three-dimensional effects in the wave stress amplitude and orientation. For clarity and convenience, a brief description is given here. On the assumption that the subgrid-scale orography has the shape of one single elliptical mountain, the mountain wave stress can be written as (*Phillips* 1984)

$$(\tau_1,\tau_2) = \rho_H U_H N_H H^2 bG(B \cos^2\psi_H + C \sin^2\psi_H , (B - C) \sin\psi \cos \psi_H) \tag{17}$$

where $B = 1 - 0.18\gamma - 0.04\gamma^2$, $C = 0.48\gamma + 0.3\gamma^2$ and G is a constant of order unity. Furthermore, when b or a are significantly smaller than the length L, characteristic of the gridpoint region size, there are, typically, $L^2/(4ab)$ ridges inside the grid-point region. Summing all the associated forces we find the stress per unit area, viz.

$$(\tau_1, \tau_2) = \rho_H U_H N_H \mu\sigma G\{B \cos^2\psi_H + C \sin^2\psi_H , (B - C) \sin\psi_H \cos\psi_H\} \tag{18}$$

where H has been replaced by 2μ, and a by μ/σ.

It is worth noting that, since the basic parameters ρ_H, U_H, N_H are evaluated for the layer between μ and 2μ above the mean orography that defines the model's lower boundary, there will be much less diurnal cycle in the stress than in previous formulations that used the lowest model levels for this evaluation. The vertical distribution of the gravity-wave stress will determine the levels at which the waves break and slow down the synoptic flow.

Since this part of the scheme is active only above the blocked flow, this stress is now constant from the bottom model level to the top of the blocked flow, $Z_{\rm blk}$. Above $Z_{\rm blk}$, up to the top of the model, the stress is constant until the waves break. This occurs when the total Richardson number, $\rm Ri$, falls below a critical value $\rm Ri_{crit}$, which is of order unity. When the non-dimensional mountain height is close to unity, this algorithm will usually predict wave breaking at relatively low levels; this is not surprising since the linear theory of mountain gravity waves predicts low-level breaking waves at large non-dimensional mountain heights (*Miles and Huppert* 1969). In reality, the depth over which gravity-wave breaking occurs is more likely to be related to the vertical wavelength of the waves. For this reason, when low-level wave breaking occurs in the scheme, the corresponding drag is distributed (above the blocked flow), over a layer of thickness Δz, equal to a quarter of the vertical wavelengths of the waves, i.e.

$$\int_{Z_{\rm blk}}^{Z_{\rm blk}+\Delta z} \frac{N}{U_{\rm p}} {\rm d}z \approx \frac{\pi}{2} \tag{19}$$

Above the height $Z_{\rm blk} + \Delta z$ are waves with an amplitude such that $\rm Ri > Ri_{crit}$.

3. SPECIFICATION OF SUBGRID-SCALE OROGRAPHY

For completeness, the following describes how the subgrid-scale orography fields were computed by *Baines and Palmer* (1990). The mean topographic height above mean sea level over the gridpoint region (GPR) is denoted by $\bar{h}$, and the coordinate z denotes elevation above this level. Then the topography relative to this height $h(x, y) - \bar{h}$ is represented by four parameters, as follows

- *(i)* The net variance, or standard deviation, μ, of $h(x, y)$ in the grid-point region. This is calculated from the US Navy data-set, or equivalent, as described by *Wallace et al.* (1983). The quantity μ gives a measure of the amplitude and 2μ approximates the physical envelope of the peaks.
- *(ii)* A parameter γ which characterizes the anisotropy of the topography within the grid-point region.
- *(iii)* An angle ψ, which denotes the angle between the direction of the low-level wind and that of the principal axis of the topography.
- *(iv)* A parameter σ which represents the mean slope within the grid-point region.

The parameters γ and ψ may be defined from the topographic gradient correlation tensor

$$H_{ij} = \overline{\frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_j}},$$

where $x_1 = x$, and $x_2 = y$, and where the terms be calculated (from the USN data-set) by using all relevant pairs of adjacent gridpoints within the grid-point region. This symmetric tensor may be diagonalized to find the directions of the principal axes and the degree of anisotropy. If

$$K = \frac{1}{2}\left\{\overline{\left(\frac{\partial h}{\partial x}\right)^2} + \overline{\left(\frac{\partial h}{\partial y}\right)^2}\right\},\ L = \frac{1}{2}\left\{\overline{\left(\frac{\partial h}{\partial x}\right)^2} - \overline{\left(\frac{\partial h}{\partial y}\right)^2}\right\} \text{ and } M = \overline{\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}}, \tag{20}$$

the principal axis of H_{ij} is oriented at an angle θ to the x-axis, where θ is given by

$$\theta = \frac{1}{2}\arctan(M/L). \tag{21}$$

This gives the direction where the topographic variations, as measured by the mean-square gradient, are largest. The corresponding direction for minimum variation is at right angles to this. Changing coordinates to x', y' which are oriented along the principal axes $x' = x \cos\theta + y \sin\theta$ and $y' = y \cos\theta - x \sin\theta$, the new values of K, L and M relative to these axes, denoted K', L' and M', are given by

$$K' = K, \quad L' = (L^2 + M^2)^{\frac{1}{2}} \quad \text{and} \quad M' = 0 \quad ,$$

where K, L and M are given by Eq. (20). The anisotropy of the orography or 'aspect ratio'. γ is then defined by the equations

$$\begin{aligned} \gamma^2 &= \overline{\left(\frac{\partial h}{\partial y'}\right)^2} \Big/ \overline{\left(\frac{\partial h}{\partial x'}\right)^2} \\ &= \frac{K' - L'}{K' + L'} = \frac{K - (L^2 + M^2)^{1/2}}{K + (L^2 + M^2)^{1/2}} \end{aligned} \tag{22}$$

If the low-level wind vector is directed at an angle φ to the x-axis, then the angle ψ is given by

$$\psi = \theta - \varphi . \tag{23}$$

The slope parameter, σ, is defined as

$$\sigma^2 = \overline{\left(\frac{\partial h}{\partial x'}\right)^2} , \tag{24}$$

i.e. the mean-square gradient along the principal axis.

APPENDIX B LIST OF SYMBOLS

a half mountain width in x-direction
B function of the mountain anisotropy
b half mountain width in y-direction
C_{d} drag coefficient
D_{blk} drag due to flow in blocked layer
G function of the mountain sharpness
H maximum mountain height ($= 2\mu$)
$h(x, y)$ mountain height profile
H_{n} non-dimensional mountain height ($= NH/|U|$)
$H_{\mathrm{n_{crit}}}$ critical non-dimensional mountain height
L length scale of the grid-point region
$l(z)$ horizontal width of mountain seen by the upstream flow
N Brunt–Väisälä frequency
N_H Brunt–Väisälä frequency of un-blocked flow evaluated at height $H(= 2\mu)$
Ri Richardson number

$\mathrm{Ri}_{\mathrm{crit}}$ critical Richardson number

U wind speed in x-direction

U_H wind speed of incident un-blocked flow evaluated at height $H(=2\mu)$

U_{p} component of the wind speed in the direction of U_H

U_τ component of wind speed in the direction of the stress τ

V wind speed in y-direction

Z_{blk} depth of blocked layer

γ anisotropy of the orography ($= a/b \le 1$)

θ orientation of the orography

μ standard deviation of orography

ρ_0 density of air at the surface

ρ_H density of the un-blocked flow evaluated at height $H(=2\mu)$

σ slope of the orography

τ_{blk} stress due to blocked flow

τ_{wave} surface stress due to gravity waves

ψ angle between incident flow and orographic principal axis

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