Calculating vertical motion using Richardson's equation

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1. Introduction

This work is part of a project to improve the assimilation of moisture in the Met Office's variational data assimilation scheme (VAR). The aim is to couple moisture and vertical motion through the product of the vertical gradient of the background moisture field, an increment to vertical velocity, and an advective timescale. In this paper we describe how we calculate suitably balanced increments to vertical motion.

The vertical component of velocity, *w*, is very difficult to observe with any accuracy. It is therefore common practice, given both the lack of observations and the assumption that an initial state will be close to hydrostatic balance, to ignore vertical motion in data assimilation schemes. In the Met Office's Unified Model (UM), which has a non-hydrostatic dynamical core, vertical velocity increments are not added to the background field at the start of a forecast and therefore the analysed vertical motion is simply the first guess field. The forecast model is updated with increments from the hydrostatic 3D-VAR data assimilation scheme, and vertical motion consistent with the new state is then generated by an explicit adjustment, or 'spin-up', of the model. However, it is possible that benefits are to be gained from explicitly calculating vertical motion within an assimilation routine:

i) the need for 'spin-up' could be reduced, permitting balanced fields to exist from an earlier stage in the forecast run; and

ii) these increments could be used in an analysis of moisture, provided that a relationship between the two could be established (see, for example, Nuret *et al.* 2000).

The continuity equation, or some approximation to it, is often used to diagnose vertical velocity in hydrostatic primitive equation models. Two desirable qualities of any method used to diagnose this variable within a data assimilation scheme are that the vertical motion should be both suitably balanced and consistent with the dynamical and thermodynamical equations. These conditions are not necessarily met by integrating the continuity equation.

A solution of the continuity equation alone need not also be a solution of the full equations of motion. That is, the values of vertical velocity thus obtained may not be consistent with the horizontal momentum balance and thermodynamics (for a discussion of kinematical and dynamical motions, see Roulstone and Sewell 1996). This may pose particular problems in data assimilation. Further, the existence of two boundary conditions on *w*, one each at the base and at the top of the model, means that the integration of the continuity equation (a first order partial differential equation) is an over-determined problem. In order to deal with this constraint, a second order equation is required. One such equation is Richardson's equation (Richardson 1922). In height co-ordinates this is

$$\gamma p \frac{\partial w}{\partial z} = \gamma p \left[\frac{Q}{Tc_p} - \nabla_z . \underline{v} \right] - \underline{v} . \nabla_z p + g \int_z^{\infty} \nabla_h . \left(\rho \underline{v} \right) dh \qquad (1)$$

Standard notation is used: Q is the diabatic heating rate per unit mass and the use of the subscript z indicates that gradient operators are evaluated on a horizontal plane. The horizontal wind is $\underline{v}$, ρ is the density, p is the pressure, γ is the ratio of specific heats, C_p is the specific heat at constant pressure, g is the acceleration due to gravity and T is temperature. This equation can be written as a second order equation for $w(z)$, which removes the integral in (1).

Equation (1) determines the vertical gradient of w at height z in terms of the other variables at greater heights and is derived by combining the continuity equation, the hydrostatic equation and the thermodynamic equation (White, 2002). A different treatment is necessary if an upper boundary condition is applied at a finite height (Kasahara and Washington 1967).

2. A scheme to calculate *w* from Richardson's equation

As a first step in our study, the diabatic heating term, Q, in (1), is set to zero. Although we anticipate that this source/sink term will be important in certain circumstances, we note that a coupling between the inviscid dynamics and the thermodynamical fields, assuming conservation of entropy, is implicit in Richardson's equation (White 2002). With this approximation in mind, (1) can be written in finite difference form for a particular model level m in the atmosphere as

$$\gamma p_m \left.\frac{\partial w}{\partial z}\right|_m = -\gamma p_m \nabla_z . \underline{v_m} - \underline{v_m} . \nabla_z p_m + g \sum_{i=m}^{i(TOA)} \left[\nabla_z . \left(\rho_i \underline{v_i} \right) \right] \Delta z_i \tag{2}$$

where $i(TOA)$ is the model level number taken to be at the top of the atmosphere and Δz_i is the vertical distance between model levels i and i-1. The density can be removed from the third term on the right hand side by recourse to the hydrostatic equation and, in order to set up an algorithm suitable for use in incremental variational analysis, (2) is then linearised about a background state. After some rearrangement, the following approximate equation for the vertical gradient of the vertical velocity increment is obtained

$$\begin{aligned} \frac{\partial w_m'}{\partial z} \approx \frac{1}{\gamma \overline{p_m}} \Bigg\{ & -\gamma p_m' \frac{\partial \overline{w_m}}{\partial z} - \gamma \overline{p_m} \nabla_z . \underline{v_m}' - \gamma p_m' \nabla_z . \overline{\underline{v_m}} - \overline{\underline{v_m}} . \nabla_z p_m' - \underline{v_m}' . \nabla_z \overline{p_m} \\ & - \sum_{i=m}^{i(TOA)} \Bigg[\overline{u_i} \frac{\partial}{\partial x} \left(\left. \frac{\partial p'}{\partial z} \right|_i \right) + u_i' \frac{\partial}{\partial x} \left(\left. \frac{\partial \overline{p}}{\partial z} \right|_i \right) + \overline{v_i} \frac{\partial}{\partial y} \left(\left. \frac{\partial p'}{\partial z} \right|_i \right) + v_i' \frac{\partial}{\partial y} \left(\left. \frac{\partial \overline{p}}{\partial z} \right|_i \right) \\ & + \left. \frac{\partial \overline{p}}{\partial z} \right|_i \nabla_z . \underline{v_i}' + \left. \frac{\partial p'}{\partial z} \right|_i \nabla_z . \overline{\underline{v_i}} \Bigg] \Delta z_i \Bigg\} \end{aligned} \tag{3}$$

Here, the overbar terms represent components of the background state and the primed quantities are the increments to those components. Note that the calculations for the top levels must be done first. In writing the code for this procedure, it is necessary to step downwards and we assume that the vertical gradient of w' vanishes at the top model surface.

Nearly all the derivatives are approximated by means of centred differencing, largely making use of the staggered nature of the UM New Dynamics (ND) grid structure. In the horizontal plane, an Arakawa C-grid is used. This is illustrated in Figure 1, which shows how derivatives of a particular variable, calculated between adjacent grid points, are evaluated at an intermediate grid point. In the vertical direction, the Charney-Phillips grid is used. This is shown in Figure 2. The vertical gradient of pressure is calculated at the lowest level by a one-sided differencing technique involving values of pressure at the lowest two levels in the model.

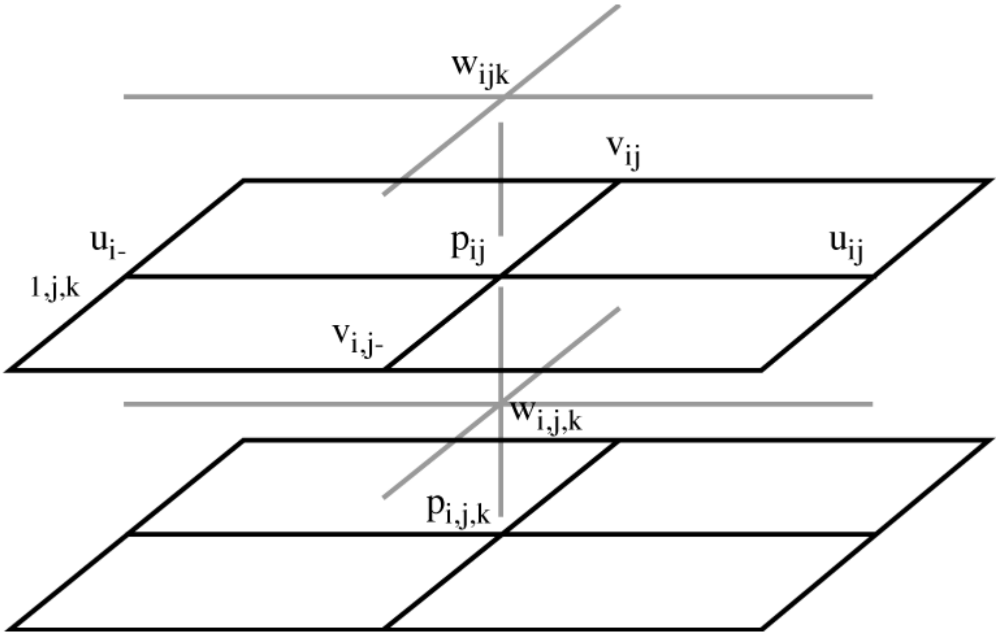

***Figure 1**: Variables held in the UM on the Arakawa 'C' grid: i (longitude), j (latitude) and k (vertical). The variables are u (horizontal east-west component of wind), v (horizontal north-south component of wind), w (vertical component of wind) and p (pressure).*

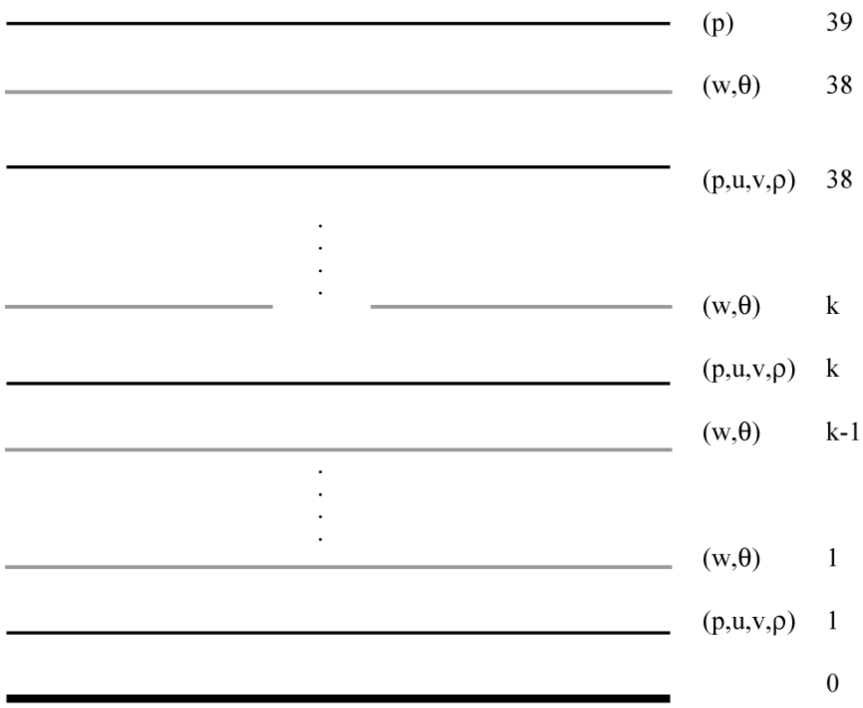

***Figure 2**: Variables held in the UM on the Charney-Phillips grid. Model level numbers are shown on the right with variables held on the levels shown. Variables are as in Figure 1 plus θ (potential temperature) and ρ (density).*

Linear interpolation is used in order to obtain the values of variables at locations required in the calculation. Furthermore, since pressure and vertical velocity are held on separate levels in the ND version of the UM, it is also necessary to interpolate the final values for the vertical gradient of w' to pressure levels. Thereafter, w' itself can be calculated on the levels at which the background values of w are also held. This last calculation is performed by assuming that w' is zero at the Earth's surface, then stepping up through the atmosphere from below, multiplying the gradient in each model layer by the thickness of that layer and summing the result. That is

$$w_m' = w_{m-1}' + \left.\frac{\partial w'}{\partial z}\right|_m \Delta z_m \tag{4}$$

In the above procedure, no account is taken of the fact that model surfaces are not necessarily horizontal. However, in the UM, the lower levels are terrain-following. Therefore, near orography, the calculation of 'horizontal' derivatives is likely to be in error if simply the values of a parameter on one model level are used. Nevertheless, by inspection of (3), it is apparent that the only place where this might be significant is in the fifth term in the bracket on the right-hand side, where the horizontal gradients of the background pressure field appear. What is actually required here is the gradient of the pressure along a horizontal surface (z) instead of along a model level surface (η). This is obtained by making a standard co-ordinate transformation and using the hydrostatic equation to replace the vertical derivative of pressure, so that

$$\begin{aligned}\left(\left[\frac{\partial \overline{p}}{\partial x}\right]_z, \left[\frac{\partial \overline{p}}{\partial y}\right]_z\right) &= \left(\left[\frac{\partial \overline{p}}{\partial x}\right]_\eta, \left[\frac{\partial \overline{p}}{\partial y}\right]_\eta\right) - \frac{\partial p}{\partial z}\left(\left[\frac{\partial z}{\partial x}\right]_\eta, \left[\frac{\partial z}{\partial y}\right]_\eta\right) \\ &= \left(\left[\frac{\partial \overline{p}}{\partial x}\right]_\eta, \left[\frac{\partial \overline{p}}{\partial y}\right]_\eta\right) + \rho g\left(\left[\frac{\partial z}{\partial x}\right]_\eta, \left[\frac{\partial z}{\partial y}\right]_\eta\right) \\ &\approx \left(\frac{p_{i+1}^\eta - p_i^\eta}{\Delta x} + \frac{\left(\rho_{i+1}^\eta + \rho_i^\eta\right)}{2} g \frac{\left(z_{i+1} - z_i\right)}{\Delta x}, \frac{p_{j+1}^\eta - p_j^\eta}{\Delta y} + \frac{\left(\rho_{j+1}^\eta + \rho_j^\eta\right)}{2} g \frac{\left(z_{j+1} - z_j\right)}{\Delta y}\right),\end{aligned}$$

where the final approximate equality shows the discretisation techniques in the $\boldsymbol{i}$ (East-West) and $\boldsymbol{j}$ (North-South) directions. (To simplify the appearance of this expression, only the relevant co-ordinate subscripts are shown.)

3. Testing the scheme

Due to the lack of observations of vertical velocity, it is extremely difficult to set up a true test of any routine which aims to diagnose it. Nevertheless, two methods of investigating whether the output is sensible and consistent have been developed and are described here.

3.1. Forecast difference tests

Output from two separate runs of the UM is taken, such that the twelve hour forecast from the first run is valid at the same time as the six hour forecast from the second run. These two forecast fields are then subtracted from each other, which creates a set of forecast difference fields; one for each model variable in the forecasts. These differences will resemble the increments produced by VAR. The Richardson scheme described in Section 2 is then applied to the difference fields for horizontal winds and pressure, in order to derive a vertical velocity difference field, which is then compared to the one generated by differencing the full model fields of w.

Figure 3 shows the results of this experiment for a model level 20 (~500 hPa) using UM forecasts valid at 1200 UTC on 26th December 1999. It is clear that there is good agreement between the directly obtained difference field and that obtained using the Richardson scheme. However, making the diagnosis by using a discretisation of the continuity equation, $\nabla.\underline{U} = 0$ (where $\underline{U}$ is the three-dimensional wind field), is obviously less successful.

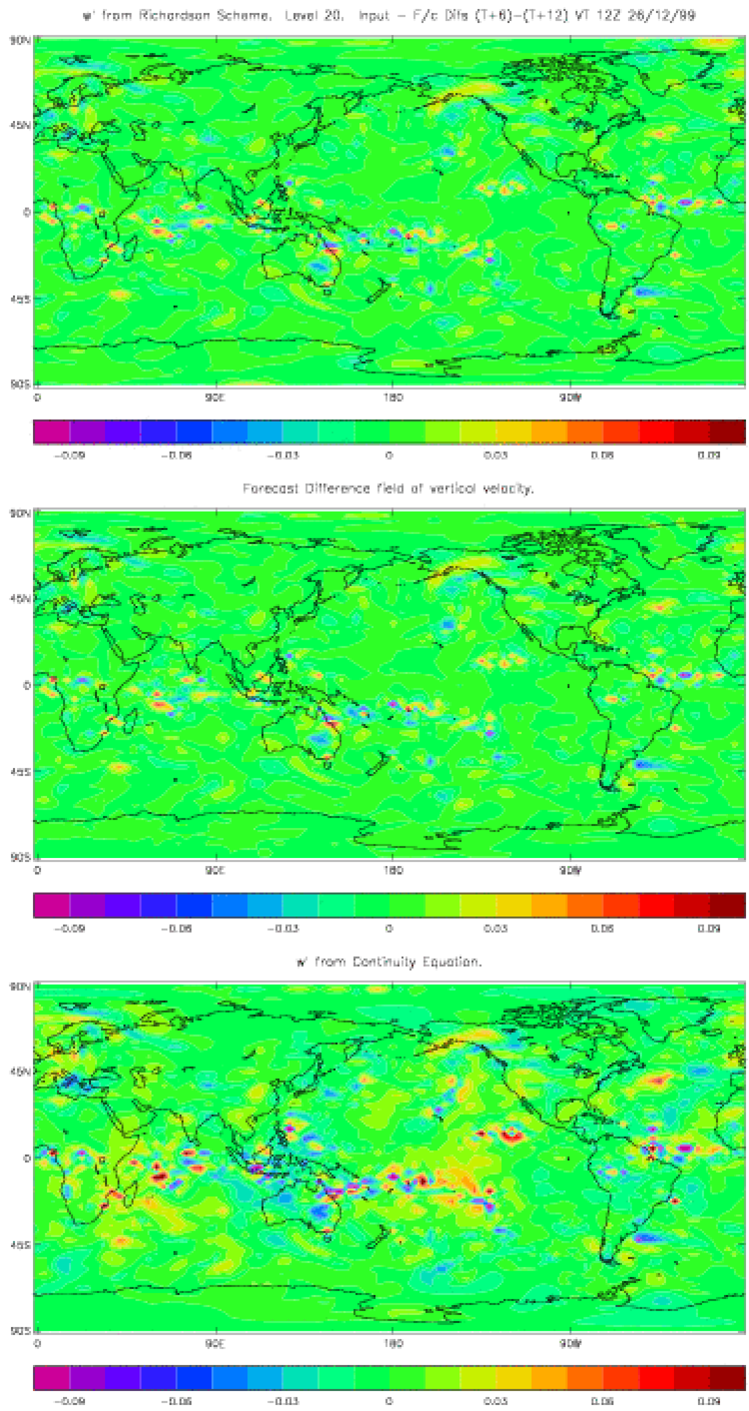

Figure 3 The colours show the distribution of mid-tropospheric vertical velocity increments (ms^{-1}).

3.2. Tests with real analysis increments

A further experiment was carried out, in which the real analysis increments generated by 3D-VAR were supplied to the Richardson scheme. This time, there was no clear cut means of verifying the results. It would have been desirable to compare the increments produced by the scheme with the differences between the actual updated analysis and original background fields. However, because of the operation of the

Incremental Analysis Update (IAU) system in 3D-VAR, the T+0 analysis is never actually available and therefore, this could not be done. Instead, a subjective analysis of the results had to be carried out with the aim of determining whether or not the Richardson scheme produced vertical velocity increments which were reasonable and consistent with the prevailing meteorological situation.

Figure 4 shows the vertical velocity increments diagnosed by the Richardson scheme and the coincident mean sea level pressure pattern over North-West Europe. Areas of enhanced ascent and descent over northern Germany and France respectively are clearly associated with the eastwards passage of a major depression. Of greater significance however, is the area just West of the Brest Peninsula where a developing trough has clearly not been well captured by the model and consequently, a requirement to increase the rate of ascent within it has been identified in the assimilation routine and diagnosed by the Richardson scheme. Study of the observations over southern UK and north-western France in the hours following, reveal marked pressure falls and a widespread area of moderate or heavy rain moving eastwards. This lends weight to the idea that there had been something of a deficiency in the model background field at the start of the assimilation, which might have been rectified by inclusion of the diagnosed vertical velocity increments.

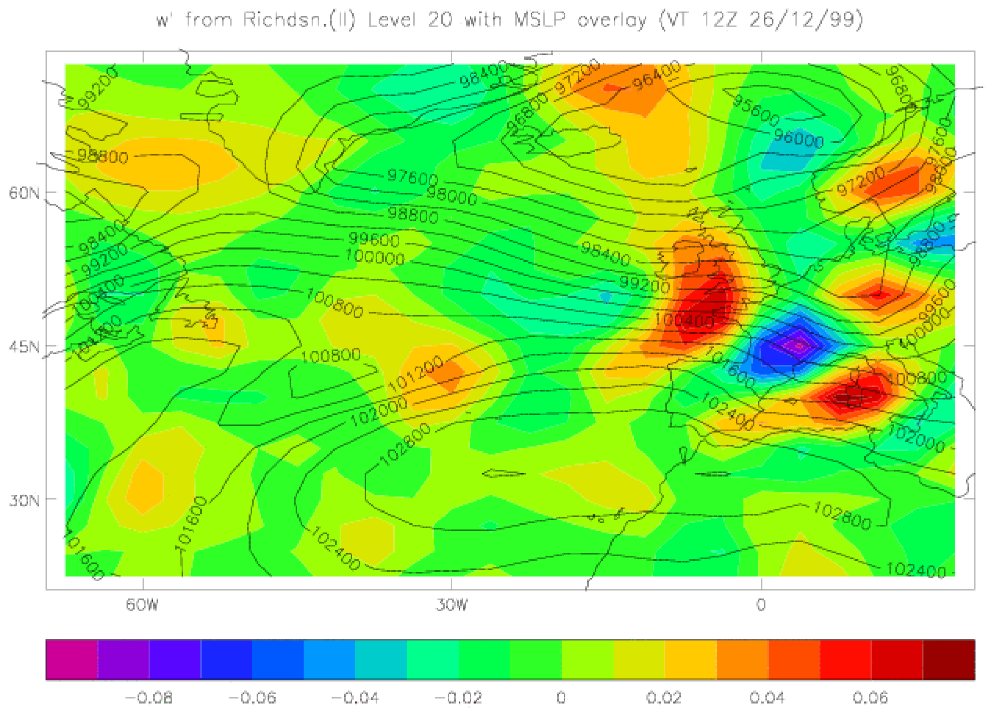

Figure 4:*1200 UTC on 26th December 1999. The colours show the distribution of mid-tropospheric vertical velocity increments (ms^{-1}), at level 20, generated from the Richardson scheme. Overlaid (black contours) is the mean sea level pressure pattern (Pa) from the model background field used in the assimilation run.*

Observations also show that the trough system extended northwards as far as Wales and Figure 5 reveals that, here too, there is a deficiency in the model background, this time in the field of relative humidity. Overcast conditions from about 10000 feet were being reported, whereas the model relative humidity field is clearly nowhere near saturated at these levels. It is encouraging to note therefore, that the vertical velocity increments are positive in this region, since it seems quite reasonable that enhanced ascent should lead to an increase in analysed relative humidity over this area. In turn, one would hope that this would lead to an improved forecast for the weather associated with the trough. It did indeed rain quite heavily over southern Wales that afternoon.

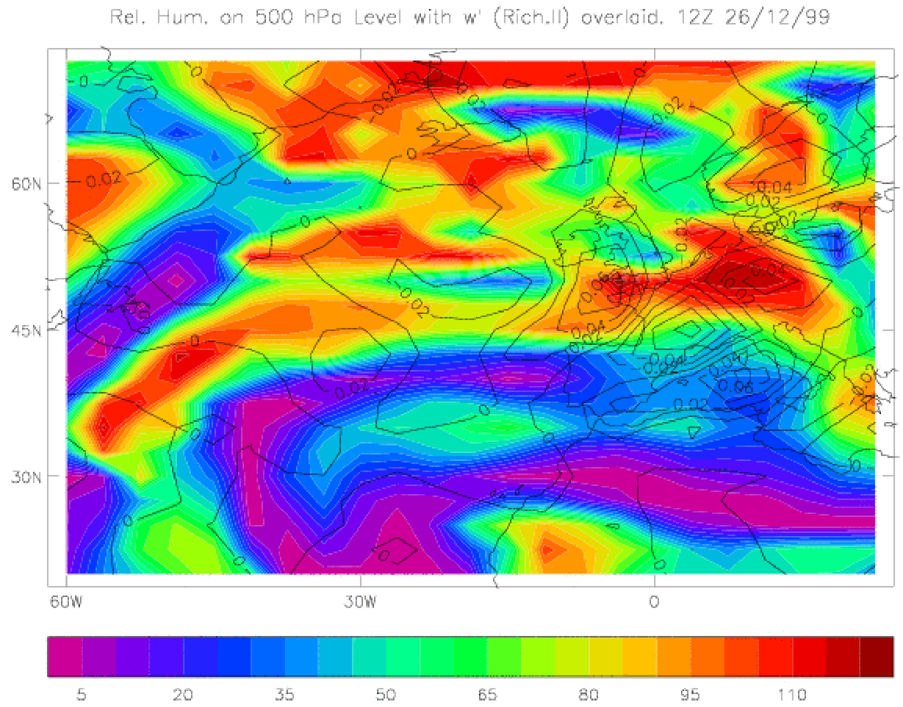

***Figure 5**: 1200 UTC on 26th December 1999. The colours show the relative humidity in the model background in the mid-troposphere. Overlaid (black contours) is the pattern of vertical velocity increments (ms^{-1}) as in Figure 4.*

4. Discussion

This investigation is obviously at a very early stage. Nevertheless, the results obtained so far seem to justify a reasonable level of confidence in applying the Richardson scheme to calculating increments for use in VAR. However, using it as a first step towards improving moisture assimilation is not going to be easy, as the relationship between humidity and vertical velocity can be subtle. For example, some vertical moisture transport occurs on a sub-grid scale and furthermore, air detrained by convection may persist in its properties long after the convection itself has died away (N.B. Ingleby, personal communication).

The scheme itself may not be valid at different scales and model resolutions. Up to now, all the experiments described have taken place on the global scale with a horizontal resolution of 60km. It remains to be seen if the assumptions used in the derivation of the algorithm will remain valid when applied to mesoscale models (cf. Tapp and White 1976). The assumption of a hydrostatic atmosphere is a possible source of error, particularly if the scheme is used at finer resolutions.

Although diabatic heating has been ignored in this study, it will be important in some circumstances. For example, it plays a major role in generating vertical motion over tropical and equatorial regions and becomes increasingly important for motions on smaller scales as well (Hartmann, 1995). However, the encouraging results shown in Figure 3 reflect some coupling between the thermodynamic effects and the dynamics, which is implicit in Richardson's equation (White, 2002). The influence of model orography has been taken into account as described in Section 2. Nevertheless, the existence of sloping model levels near the ground may make the assumption that the vertical velocity increment is zero at the surface, somewhat questionable, since no account is taken of the effect of the wind 'blowing uphill'. In the absence of any definitive evidence to the contrary, this assumption (effectively a 'no-slip' boundary condition) is considered to be valid although it may become less appropriate as the model resolution is reduced.

The choice of the uppermost model level as being the top of the atmosphere may also introduce errors. In the derivation of Richardson's equation, the assumption is made that the time derivative of pressure vanishes at this level and furthermore, in this study, that the vertical gradient of the vertical velocity increment does too. An alternative approach (e.g. Kasahara and Washington, 1967) is to use zero vertical velocity as a boundary at the top model level and then determine a value for the local pressure tendency there.

Among the avenues for further research are:

i) performing new experiments with forecast difference fields at finer resolutions;

ii) investigating what effect the use of diagnosed vertical velocity increments has on the accuracy of the full forecast model;

iii) investigating the possible links between vertical velocity increments and moisture or precipitation increments at various scales;

iv) providing such links can be found, determining how to adjust the moisture or precipitation increments using the output from the Richardson scheme.

We also remark that Richardson's equation arises as a consistency condition in the hamiltonian description of the hydrostatic primitive equations (Theiss 1999). This result may have practical value because it suggests ways of imposing additional balance constraints on the momentum equations that project onto balanced vertical motions.

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