Atmospheric waves

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1. INTRODUCTION

In order to understand the many and varied approximations and assumptions which are made in designing a numerical model of the earths atmosphere (whether of small-scale features such as individual clouds, or mesoscale, regional, global weather prediction or climate models), it is necessary to study the various wave motions which can be present. The identification and appreciation of the mechanisms of these waves will allow us to isolate or eliminate certain wave types and to better understand the viability and effectiveness of commonly made approximations such as assuming hydrostatic balance.

Here we will attempt to identify the basic atmospheric wave motions, to emphasise their most important properties and their physical characteristics. Basic textbooks such as 'An Introduction to Dynamic Meteorology' by *Holton* or 'Numerical Prediction and Dynamic Meteorology' by *Haltiner and Williams* provide a good introduction and complement this short course well. Whereas these textbooks treat each wave motion separately starting from different simplified equation sets, we will separate waves from a more complete equation set by various approximations made at later stages thus making it easier to relate the impact of a particular approximation on several wave types. Nevertheless, to retain all wave types in the initial equation set is unmanageable and we will introduce subsets of equations later.

We will restrict ourselves to neutral (non-amplifying, non-decaying) waves in which energy exchanges are oscillatory.

While the exact equations have been considerably simplified into a useful form for studies of atmospheric dynamics, these simplified equations cannot be solved analytically except in certain special cases. The main difficulty arises through the non-linear advective terms $(\mathbf{v} \cdot \nabla)\mathbf{v}$, $\mathbf{v} \cdot \nabla T$ etc. Although these terms can considerably modify the linear solutions and are also physically significant because they represent transfer and feedback between different scales of motion, the linearized equations (obtained by removing second-order advective terms) are useful for identifying the origin of distinct types of wave (acoustic, gravity and cyclone) in the equations. It should be understood that although nonlinearity will modify the acoustic, gravity and long waves it will not introduce any additional wave types. Since the origin of waves can be identified in the linearized equations and these can be solved analytically, useful methods for filtering individual modes can be determined.

2. BASIC EQUATIONS

The following mathematical analysis requires a choice of vertical coordinate. While it is common to use pressure or a pressure-based vertical coordinate in large-scale modelling, this is not a necessary theoretical restriction and we can write the equations in z (height), p (pressure) or $\sigma\,(= p/p_*)$, as follows:

***z* -coordinates** (x, y, z, t)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (1)$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad (2)$$

$$\frac{Dw}{Dt} + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{D}{Dt}(\ln\rho) \quad (4)$$

$$\frac{D}{Dt}(\ln T) = \kappa\frac{D}{Dt}(\ln p) \quad (5)$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ and $\kappa = \frac{R}{c_p}$

All source/sink terms are omitted.

***p*-coordinates** (x, y, p, t)

An exact transformation of (1)–(5) to p-coordinates gives:

$$\frac{Du}{Dt} - fv = -(1+\varepsilon)\frac{\partial\Phi}{\partial x} \quad (6)$$

$$\frac{Dv}{Dt} + fu = -(1+\varepsilon)\frac{\partial\Phi}{\partial x} \quad (7)$$

$$\frac{RT}{p} = -(1+\varepsilon)\frac{\partial\Phi}{\partial p} \quad (8)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial\omega}{\partial p} = \frac{D}{Dt}\{\ln(1+\varepsilon)\} \quad (9)$$

$$\frac{DT}{Dt} = \frac{\kappa\omega T}{p} \quad (10)$$

where

$$\varepsilon = \frac{1}{g}\frac{Dw}{Dt} = \frac{1}{g^2}\frac{D}{Dt}\left[\frac{D\phi}{Dt}\right] \quad (11)$$

and $\omega = \frac{Dp}{Dt}$, $\Phi = gz$, and $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$

σ-coordinates $(x, y, \sigma = p/p_*, t)$

Either by transforming (1)(5) or (6)–(11) we can obtain the exact σ-coordinate set:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -(1+\varepsilon)\frac{\partial\Phi}{\partial x} - RT\frac{\partial}{\partial x}(\ln p_*) \tag{12}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu = -(1+\varepsilon)\frac{\partial\Phi}{\partial y} - RT\frac{\partial}{\partial y}(\ln p_*) \tag{13}$$

$$\frac{RT}{\sigma} = -(1+\varepsilon)\frac{\partial\Phi}{\partial\sigma} \tag{14}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial\dot{\sigma}}{\partial\sigma} = -\frac{\mathrm{D}}{\mathrm{D}t}(\ln p_*) + \frac{\mathrm{D}}{\mathrm{D}t}\{\ln(1+\varepsilon)\} \tag{15}$$

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \kappa T\left\{\frac{\dot{\sigma}}{\sigma} + \frac{\mathrm{D}}{\mathrm{D}t}(\ln p_*)\right\} \tag{16}$$

where $\dot{\sigma} = \frac{D\sigma}{Dt}$, and $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \dot{\sigma}\frac{\partial}{\partial\sigma}$.
All partial derivatives respect their coordinate system.

Note that putting $\mathrm{D}w/\mathrm{D}t = 0$ in (3), and $\varepsilon = 0$ elsewhere gives the familiar large-scale equation sets, but we will not make this approximation at present. In principle the following linearized analysis can be done in any coordinate; we will use height but much of this analysis has been done in pressure and sigma coordinates elsewhere (e.g. *Kasahara*, 1974; *Miller*, 1974; *Miller and White*, 1984).

For simplicity we will suppose that the motion is independent of y and neglect the variation of the Coriolis parameter with latitude ($\beta = \partial f/\partial y = 0$). Also we will consider small disturbances on an initially motionless atmosphere. A non-zero basic flow and restoration of β and non-zero $\partial/\partial y$ will be considered later.

In order to trace the effect of individual terms we will use 'tracer parameters' n_1, n_2, n_3, n_4 which have the value 1 but can be set to zero to eliminate the relevant term.

We define $\Theta \equiv \ln\theta$,where $\theta = T\left(\frac{p_{00}}{p}\right)^{\kappa}$ and p_{00} is a reference pressure (e.g. 10^5 Pa), and write

$$\begin{aligned}
u &= u_0 + \delta u = \delta u \\
v &= v_0 + \delta v = \delta v \\
w &= w_0 + \delta w = \delta w \\
\rho &= \rho_0(z) + \delta\rho \\
p &= p_0(z) + \delta p \\
\Theta &= \Theta_0(z) + \delta\Theta
\end{aligned}$$

where $\delta u, \delta v, \delta w, \delta\rho, \delta p, \delta\Theta$ denote small perturbations on a mean state and $\rho_0(z), p_0(z), \Theta_0(z)$ define the basic horizontally stratified atmosphere with $\frac{\partial p_0}{\partial z} = -g\rho_0$. Since we consider small perturbations such that products of perturbations can be negelcted and that $|\delta p|/p_0 \ll 1, |\delta\rho|/\rho_0 \ll 1, |\delta\Theta|/\Theta_0 \ll 1$ we can write (1)–(5) as:

$$\frac{\partial \delta u}{\partial t} - f\delta v + \frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_0}\right) = 0 \tag{17}$$

$$\frac{\partial \delta v}{\partial t} + f\delta u = 0 \tag{18}$$

$$n_4\frac{\partial \delta w}{\partial t} + \frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) - n_3 B\frac{\delta p}{\rho_0} - g\delta\Theta = 0 \tag{19}$$

$$n_2\frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho_0}\right) + \frac{\partial \delta u}{\partial x} + \frac{\partial \delta w}{\partial z} - \frac{n_1\delta w}{H_0} = 0 \tag{20}$$

$$\frac{\partial}{\partial t}\delta\Theta + \delta w B = 0 \tag{21}$$

where $B = \frac{\partial}{\partial z}(\ln\theta_0)$ and $\frac{1}{H_0} = -\frac{\partial}{\partial z}(\ln\rho_0)$.

EXERCISE:

Derive Eq. (19)

The coefficients $f,\ B,\ g,\ 1/H_0$ in (17)–(21) are independent of x and t, so these equations are linear and, in an unbounded region, admit solutions of the separable form $A(z)\exp(i(kx + \sigma t))$ where A can be a complex function, and k and σ are the x-wavenumber and the frequency respectively. Complex values of σ would imply amplifying/decaying waves which are not considered here.

The full solution is the appropriate Fourier sum of terms of this form over all wavenumbers. Since we shall be looking at individual waves we choose to discuss individual wave components rather than the Fourier sum.

Inserting $\delta u = \hat{u}(z)\exp(i(kx + \sigma t))$, $\delta v = \hat{v}(z)\exp(i(kx + \sigma t))$, and the corresponding expressions for δw, δp, $\delta\rho$, and $\delta\Theta$ into (17)-(21), and noting that the operators $\partial/\partial x$, $\partial/\partial t$ can be replaced by ik and $i\sigma$, respectively, yields the following set of ordinary differential equations in the unknowns $\hat{u}, \hat{v}, \hat{w}, \hat{p}/\rho_0, \hat{\rho}/\rho_0, \hat{\Theta}$:

$$i\sigma\hat{u} - f\hat{v} + ik\frac{\hat{p}}{\rho_0} = 0 \tag{22}$$

$$i\sigma\hat{v} + f\hat{u} = 0 \tag{23}$$

$$n_4 i\sigma\hat{w} + \frac{d}{dz}\left(\frac{\hat{p}}{\rho_0}\right) - n_3 B\frac{\hat{p}}{\rho_0} - g\hat{\Theta} = 0 \tag{24}$$

$$n_2 i\sigma\frac{\hat{\rho}}{\rho_0} + ik\hat{u} + \frac{d}{dz}\hat{w} - \frac{n_1\hat{w}}{H_0} = 0 \tag{25}$$

$$i\sigma\hat{\Theta} + \hat{w}B = 0 \tag{26}$$

Then (22) and (23) give:

$$\hat{u} = -\frac{\sigma k}{\sigma^2 - f^2}\frac{\hat{p}}{\rho_0} \tag{27}$$

$$\hat{v} = -\frac{ifk}{\sigma^2 - f^2}\frac{\hat{p}}{\rho_0} \tag{28}$$

Eliminating $\hat{u}$ between (25) and (27), and using (26) and the relation:

$$\delta\hat{\Theta} = \frac{1}{\gamma}\frac{\hat{p}}{p_0} - \frac{\hat{\rho}}{\rho_0} = \frac{1}{c^2}\frac{\hat{p}}{\rho_0} - \frac{\hat{\rho}}{\rho_0}$$

where $c = \sqrt{\gamma R T_0}$ is the Laplacian speed of sound, gives:

$$\frac{d}{dz}\hat{w} + \left(Bn_2 - \frac{n_1}{H_0}\right)\hat{w} + i\sigma\left(\frac{n_2}{c^2} - \frac{k^2}{\sigma^2 - f^2}\right)\frac{\hat{p}}{\rho_0} = 0 \tag{29}$$

Using $\hat{\Theta} = i\hat{w}\dfrac{B}{\sigma}$ (30)

to eliminate $\hat{\Theta}$ from (24) yields:

$$i\sigma\left[\frac{d}{dz} - Bn_3\right]\frac{\hat{p}}{\rho_0} + (gB - n_4\sigma^2)\hat{w} = 0 \tag{31}$$

In general B, $1/H_0$, c are functions of z, so $\hat{w}$ and $\hat{p}/\rho_0$ are obtained by simultaneous solution of the two first order equations (29) and (31) and the $\hat{u}$, $\hat{v}$, $\hat{\Theta}$, $\hat{\rho}/\rho_0$ fields obtained from (27), (28), (30) and (25), respectively. However, for our present purposes it is sufficient to consider constant (mean) values of B, $1/H_0$ and c which are related by $B + g/c^2 = 1/H_0$. Then the differential equation for the height variation of $\hat{w}$ is:

$$\sigma\left\{\frac{d^2}{dz^2} + \left[B(n_2 - n_3) - \frac{n_1}{H_0}\right]\frac{d}{dz} + (gB - n_4\sigma^2)\left(\frac{k^2}{\sigma^2 - f^2} - \frac{n_2}{c^2}\right) - Bn_3\left(Bn_2 - \frac{n_1}{H_0}\right)\right\}\hat{w} = 0 \tag{32}$$

3. Exact solutions of the linearized equations

The exact linearized equation for $\hat{w}$, obtained by setting $n_1 = n_2 = n_3 = n_4 = 1$ in (32), is:

$$\sigma\left\{\frac{d^2}{dz^2} - \frac{1}{H_0}\frac{d}{dz} + (gB - \sigma^2)\left(\frac{k^2}{\sigma^2 - f^2} - \frac{1}{c^2}\right) - B\left(B - \frac{1}{H_0}\right)\right\}\hat{w} = 0 \tag{33}$$

One solution of (33) is $\sigma = 0$, and the corresponding dynamical structure can be determined by setting $\sigma = 0$ in (22)–(26) to give $\hat{u} = \hat{w} = 0$ and $\hat{v} = (ik/f)(\hat{p}/\rho_0)$. This is geostrophic motion in the y-direction and is also exactly hydrostatic. More detail of this solution will be considered later.

If $\sigma \neq 0$ in (33) then:

$$\frac{d^2}{dz^2}\hat{w} - \frac{1}{H_0}\frac{d}{dz}\hat{w} + \left\{\frac{k^2(gB-\sigma^2)}{\sigma^2 - f^2} + \frac{\sigma^2}{c^2}\right\}\hat{w} = 0 \tag{34}$$

Setting $\hat{w} = w_*(z)\exp(z/2H_0)$ leads to the differential equation:

$$\frac{d^2 w_*}{dz^2} + \left\{\frac{k^2(gB-\sigma^2)}{\sigma^2 - f^2} + \frac{\sigma^2}{c^2} - \frac{1}{4H_0^2}\right\}w_* = 0 \tag{35}$$

In general, if the wave motion is confined between physical boundaries, then (35) must be solved with respect to boundary conditions—that is w_* must satisfy the differential equation (35) and the prescribed values of w_* on these boundaries. It is reasonable to suppose that only waves of a certain frequency and wavenumber can satisfy all these constraints. For such waves the permissible values of the (characteristic) function

$$C(k, \sigma, gB, f, c, H_0) = \frac{k^2(gB-\sigma^2)}{\sigma^2 - f^2} + \frac{\sigma^2}{c^2} - \frac{1}{4H_0^2}$$

are called eigenvalues and the solutions corresponding to these eigenvalues are called eigenfunctions.

For simplicity we shall suppose that the fluid is unbounded (the bounded problem with $w_* = 0$ at $z = \pm H/2$ (say) can be attempted as an exercise) and in this case it is readily seen that $w_* \propto \mathrm{e}^{imz}$ is a solution to (35) provided that:

$$m^2 = \frac{k^2(gB-\sigma^2)}{\sigma^2 - f^2} + \frac{\sigma^2}{c^2} - \frac{1}{4H_0^2} \tag{36}$$

This frequency or 'dispersion' relationship is of fourth order in σ:

$$\sigma^4 - \sigma^2\left[f^2 + c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right)\right] + c^2\left[k^2 gB + f^2\left(m^2 + \frac{1}{4H_0^2}\right)\right] = 0 \tag{37}$$

and contains two pairs of waves moving in the $+x$ and $-x$ directions, with:

$$\sigma_g^2 = \frac{1}{2}\left[f^2 + c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right)\right]\left[1 - \left\{1 - \frac{4c^2\left[k^2 gB + f^2\left(m^2 + \frac{1}{4H_0^2}\right)\right]}{\left[f^2 + c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right)\right]^2}\right\}^{\frac{1}{2}}\right] \tag{38}$$

$$\sigma_a^2 = \frac{1}{2}\left[f^2 + c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right)\right]\left[1 + \left\{1 - \frac{4c^2\left[k^2 gB + f^2\left(m^2 + \frac{1}{4H_0^2}\right)\right]}{\left[f^2 + c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right)\right]^2}\right\}^{\frac{1}{2}}\right] \tag{39}$$

The first pair of roots represent inertial–gravity waves and the second pair represent acoustic waves. This is not obvious, especially with the degree of complication represented in these two relationships, so it is useful to look at extreme cases (i.e. short/long waves) to clarify the above classification.

In such dispersion relations a basic unsheared zonal flow ($\bar{u}$) can be included by replacing σ by ($\sigma + k\bar{u}$), i.e. Doppler shifted.

3.1 Gravity Waves

In the troposphere, the inertial frequency f is small compared with the Brunt–Väisälä frequency $\sqrt{gB}$ (a global average of $f/\sqrt{gB} \sim 10^{-2}$). Moreover their magnitudes together with those of c and H_0 satisfy the following inequalities:

$$\frac{H_0^2 f^2}{c^2} \ll \frac{f^2}{gB} \ll \frac{gBH_0^2}{c^2} \ll 1 \tag{40}$$

Using relation (40) and considering short waves such that $k^2 \gg (gB/c^2)$, then (38) reduces to:

$$\sigma_g^2 \cong \frac{gBk^2}{(k^2 + m^2 + 1/4H_0^2)} \tag{41}$$

This is the dispersion relation for short internal gravity waves (wavelength ≤ 50 km say). Since the waves are relatively short they are not modified significantly by rotation (no f in (41)).

(41) defines an upper limit to the frequency, i.e. $\sigma_g^2 \sim gB$ for $k^2 \gg m^2$, with a period of about 10 minutes.

Since $k^2 \gg m^2$, the propagation is primarily in the vertical $(\sigma^2/k^2 \ll \sigma^2/m^2)$, with propagation speeds of approximately 10m/s.

We can substitute $\sigma_g = \sqrt{gB}$ back into (22)–(26) and obtain the 'geometry'.

EXERCISE:

Show that these oscillations are transverse (particle paths parallel to wave fronts). Derive from (41) a relationship between group velocity and phase velocity and consider its geometric interpretation.

Gravity waves act as a signal to the surrounding fluid of localised changes in potential temperature. Short gravity waves are dispersive since their phase velocity σ/k is a function of k and they radiate energy.

Internal gravity waves are excited by local diabatic or mechanical forcing. A particularly interesting, important, and ubiquitous type of gravity wave is generated when there is stably stratified flow over orography. These are know as lee waves, a stationary wave pattern over and downstream of the obstacle. Relative to the mean wind $\bar{u}$ the phase lines slope upstream; energy must be propagated upward hence the phase velocity must have a downward component (this follows from the previous exercise). Since the waves are stationary $c_x = 0$:

$$\bar{u} = \frac{\sqrt{gB}}{(k^2 + m^2 + 1/4H_0^2)^{1/2}}.$$

This can be solved for m (given $k, \bar{u}, B$) to give the phase slopes. For vertical propagation $m^2 > 0$, i.e. $\bar{u} < \sqrt{gB}/k$, hence favourable lee wave conditions require suitable local combinations of mountain/hill geometry and large-scale flow.

The importance of vertically propagating gravity waves, especially lee waves, in forecast and climate models has attracted recent attention and their associated vertical momentum fluxes appear to be a significant factor in the dynamics of the large-scale circulation and of the stratosphere in particular.

As we move towards the long wave limit of (38) the influence of rotation (through f) is increasingly felt and these long inertial gravity waves have frequencies tending towards f. Their horizontal phase speeds become large, of relevance in the design of time-integration schemes and in the problem of initialisation. The limiting case of $\sigma = f$ represents a pure inertial wave with neither buoyancy or pressure forces important.

EXERCISE:

Substitute $\sigma = f$ back into (22)–(26).

3.2 Acoustic waves

In a similar manner we can examine (39). The inequalities in (40) are independent of the character of the oscillation and considering short waves such that $k^2 » gB/c^2$ it follows that:

$$\sigma_a^2 \cong c^2(k^2 + m^2 + 1/4H_0^2) \tag{42}$$

Also if $k^2 » (m^2 + 1/4H_0^2)$, then:

$$\sigma_a^2 \cong c^2 k^2 \tag{43}$$

The structure of these short waves may be found by substituting Eq. (43) back into (22)–(26).

EXERCISE:

Show that these waves are longitudinal with negligible temperature changes.

The motion transmits pressure perturbations with a speed c, the speed of sound, for all wavelengths—i.e. the motion is nondispersive.

For long acoustic waves i.e. $\sigma^2 » f^2$, $\sigma^2 » gB$ and $k \to 0$ it is readily deduced that:

$$\sigma^2 \cong c^2(m^2 + 1/4H_0^2) \tag{44}$$

Since σ/k, σ/m are horizontal and vertical phase speeds, respectively, we can deduce that short acoustic waves propagate with wave fronts almost vertical and vice-versa for long waves.

4. SIMPLIFIED SOLUTIONS TO THE LINEARIZED EQUATIONS—FILTERING APPROXIMATIONS

It is useful to simplify the exact linearized equations since we can then extend the physical principles behind these approximations to simplify the much more complicated non-linear equations.

In particular, in this section we shall show how sound waves and gravity waves can be filtered, and determine conditions under which the static approximation to the pressure field is valid.

It is not obvious ‘a priori' how an approximation made in one equation feeds through to affect terms in other equations and hence modify the mathematical and physical solutions. We therefore carry out the necessary elimination rather carefully and for this purpose we retain the parameters n_1, n_2, n_3, n_4 to trace the effects of an approximation. However, we see from (32) that n_2 and n_3 occur in the combination $(n_2 - n_3)$ which vanishes in the exact equations $(n_1 = n_2 = n_3 = n_4 = 1)$. Consequently, a spurious term will arise if we neglect n_2 but retain n_3, or vice-versa. We therefore must either retain both n_2 and n_3 or neglect both. We therefore have to set $n_3 = n_2$ in (32).

For reasons already discussed, the acoustic and inertial-gravity solutions to (32) are proportional to $\exp\left(im + \frac{n_1}{2H_0}\right)z$, provided that the oscillations obey the following frequency equation:

$$k^2 + m^2 + \frac{n_1^2}{4H_0^2} + \frac{k^2[f^2 - gB + \sigma^2(n_4 - 1)]}{\sigma^2 - f^2} - n_2 n_4 \frac{\sigma^2}{c^2} + Bn_2\left[\frac{(1-n_1)}{H_0} + B(n_3 - 1)\right] = 0 \tag{45}$$

4.1 The elimination of acoustic waves

It is helpful to discuss the physical origin of acoustic waves in order to indicate the mathematical approximations necessary for their elimination. Acoustic waves will occur in any elastic medium, and the elastic compressibility of a fluid is represented by $\partial\rho/\partial t$ in the continuity equation, so it is reasonable to suppose that the removal of this term will filter acoustic waves. Moreover, since we want to use these simplified (anelastic) equations in, for example, a later analysis of gravity waves, we hope that the effect of filtering sound waves will not distort the gravity waves in the anelastic equations

So setting $n_3 = n_2 = 0$, $n_1 = n_4 = 1$ in the general frequency equation (45) gives:

$$\frac{k^2(gB - \sigma^2)}{\sigma^2 - f^2} = m^2 + \frac{1}{4H_0^2} \tag{46}$$

(46) has only two roots in σ, contrasting with the corresponding exact frequency equation (36) which has four roots, representing two inertial-gravity waves and two acoustic waves. Rearranging (46) and noting that $f^2/gB \ll 1$,

$$\sigma^2 = f^2 + \frac{gBk^2}{k^2 + m^2 + \frac{1}{4H_0^2}} \tag{47}$$

The short-wave approximation ($k^2 \gg (f^2/gB)(k^2 + 1/4H_0^2)$) to (47) is:

$$\sigma^2 = \frac{gBk^2}{k^2 + m^2 + \frac{1}{4H_0^2}} \tag{48}$$

which by comparison with the short-wave approximation to the exact equation (41) clearly represents an (undistorted) gravity-wave oscillation. Similarly the long inertial waves represented by $\sigma^2 \sim f^2$ are undistorted.

The basic criterion for neglect of $\partial/\partial t(\delta\rho/\rho_0)$, viz $n_2 = 0$, is obtained by comparing (46) and (36) and this criterion can be seen to be that:

$$\sigma^2 \ll c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right)$$

That is, that the frequency of inertial-gravity waves must be much smaller than the acoustic frequency, a condition always well satisfied.

From these considerations we can use the acoustically filtered equations with confidence in any detailed examination of gravity waves in the atmosphere.

Putting $n_2 = n_3 = 0$ in the fully non-linear equations defines a set of equations which do not support acoustic waves. Although local elastic density changes are absent, variations in density are included through the vertical density variation (H_0) and where multiplied by g in the vertical momentum equation.

4.2 The hydrostatic approximation

The hydrostatic approximation to the pressure field ($Dw/Dt = 0$) can be made if $\partial(\delta w)/\partial t \ll g\delta\Theta$ in the vertical component of the momentum equation (19). However, neither the precise circumstances under which this approximation is valid nor its dynamical feedback on wave structure is obvious.

Referring to (45) it is clear that the criteria for the neglect of the terms involving n_4 are:

(a) $\sigma^2 \ll c^2(k^2 + m^2 + 1/4H_0^2)$

(b) $\sigma^2 \ll gB$

The first criterion is always well satisfied by inertial-gravity and geostrophic waves, since these are of low frequency by comparison with the acoustic oscillations. The second criterion is well satisfied by inertial waves but not by very short gravity waves, since from (41) we established that $\sqrt{gB}$ is their approximate frequency. We therefore examine this second criterion a little more carefully. From (41) we see that the frequency of pure gravity waves is given by:

$$\sigma^2 \cong \frac{gBk^2}{k^2 + m^2 + 1/4H_0^2}$$

in which case $\sigma^2 \ll gB$ only if $k^2 \ll m^2 + 1/4H_0^2$. We can therefore only set $n_4 = 0$ for systems with flow 'geometry' satisfying this criterion, i.e. the horizontal wavelength must be much greater than the vertical wavelength.

To demonstrate this more clearly we set $n_4 = 0$, $n_1 = n_2 = n_3 = 1$ in (45), showing that:

$$\sigma^2 = f^2 + \frac{gBk^2}{m^2 + 1/4H_0^2} \tag{49}$$

Comparing this equation with (41) it is obvious that unless $k^2 \ll m^2 + 1/4H_0^2$, the gravity wave will be considerably distorted if the pressure field is hydrostatic. The inertial wave is, however, unaffected. So in particular, if the system being analysed has comparable vertical and horizontal wavelength, the hydrostatic assumption should not be used - for example in convective scale models. If however $L_x \gg L_z$, that is $L_x \geq 100$ km for disturbances of depth ~ 10 km, the hydrostatic approximation will not appreciably distort the gravity waves.

Acoustic frequencies do not appear in (49) and at first sight it would appear that all acoustic oscillations have been filtered from the equations of motion by making the pressure fields hydrostatic. This is not the case however, since oscillations characterised by $\hat{w} = 0$ everywhere are not represented in (49); since this equation was obtained from (32) by assuming that $\hat{w}$ was not identically zero. Vertically propagating acoustic waves ($\hat{w} \neq 0$)are, however, filtered by demanding that the pressure field should by hydrostatic.

4.3 The Lamb wave

We now examine the case where $\hat{w} = 0$ everywhere. In this case (32) is clearly redundant, and we must refer back to (29) and (31) to obtain the frequency of possible oscillations. From these equations with $\hat{w} = 0$

$$\sigma\left(\frac{n_2}{c^2} - \frac{k^2}{\sigma^2 - f^2}\right)\frac{\hat{p}}{\rho_0} = 0 \tag{50}$$

$$\sigma\left(\frac{d}{dz} - Bn_3\right)\frac{\hat{p}}{\rho_0} = 0 \tag{51}$$

Now $\hat{p}/\rho_0 = 0$ is a trivial solution because with $\hat{w} = 0$, it follows that the fluid must be motionless and hydrostatic $(\delta u = \delta v = \delta w = \delta\rho = \delta\Theta = \delta p = 0)$ i.e. the initial state.

The root $\sigma = 0$ is the geostrophic mode previously examined.

The remaining oscillation is represented by:

$$\frac{k^2}{\sigma^2 - f^2} = \frac{n_2}{c^2} \tag{52}$$

with $$\left(\frac{d}{dz} - Bn_3\right)\frac{\hat{p}}{\rho_0} = 0$$

Since n_2, n_3 trace the elastic compressibility this type of motion is a form of acoustic wave, and can be eliminated by setting $\partial\delta\rho/\partial t = 0$ in the continuity equation and $B\delta p/\rho = 0$ in the vertical momentum equation (19).

Note that, if $n_2 = 0$ but $n_3 = 1$, a spurious pressure perturbation $\hat{p}/\rho \propto e^{Bz}$ will arise.

To discuss the structure of the oscillation more carefully set $n_2 = n_3 = 1$, in which case:

$$\sigma^2 = f^2 + k^2c^2 \tag{53}$$

$$\frac{\hat{p}}{\rho_0} = \mathrm{e}^{Bz}\cos(kx + \sigma t)$$

Referring back to the equations of motion in the usual way we obtain for the rest of the variables

$$\hat{v} = \hat{w} = 0$$
$$\hat{\Theta} = 0$$
$$\hat{u} = -\frac{\sigma}{kc^2}\left(\frac{\hat{p}}{\rho_0}\right)$$

The oscillation is therefore a pressure pulse propagating horizontally at the speed of sound. This type of oscillation is known as a Lamb wave; it has negligible physical significance and is removed by the anelastic approximation. The Lamb wave can also be eliminated by appropriate upper- or lower-boundary conditions. For large-scale forecast models these methods of elimination are not usually appropriate. However the longest gravity waves have comparable phase speeds to the Lamb wave (see Section 7).

It is worth stressing that the presence of the elastic term in the continuity equation is rather concealed in the pressure- and sigma-coordinate forms of the continuity equation, however both (9) and (15) are elastic (even with the hydrostatic approximation, $\varepsilon = 0$).

Although the hydrostatic approximation filters vertically propagating sound waves it is unnecessarily restrictive and it has been shown that a small approximation to the vertical acceleration can achieve the same effect (*Miller*, 1974; *Miller and White*, 1984).

4.4 Filtering of gravity waves

For the theoretical study of large-scale dynamics and also for the earlier large-scale numerical models, the presence of gravity waves is of little significance (and a nuisance!). The following analysis demonstrates that requiring the local rate of change of divergence to be zero is a sufficient filter.

Taking (17)–(21), setting $n_1 = n_2 = n_3 = n_4 = 0$ and eliminating $\delta\Theta$ and δw leads to the equations:

$$n_5\frac{\partial}{\partial t}\left(\frac{\partial \delta u}{\partial x}\right) - f\frac{\partial \delta v}{\partial x} + \frac{\partial^2}{\partial x^2}\left(\frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \delta v}{\partial x}\right) + f\left(\frac{\partial \delta u}{\partial x}\right) = 0$$

$$\frac{\partial}{\partial t}\left\{\frac{\partial^2}{\partial z^2}\left(\frac{\delta p}{\rho_0}\right)\right\} - gB\left(\frac{\partial \delta u}{\partial x}\right) = 0$$

where we have introduced an additional tracer, n_5, on $\frac{\partial}{\partial t}\left(\frac{\partial \delta u}{\partial x}\right)$. These equations have the dispersion relation:

$$\sigma\left[(f^2 - n_5\sigma^2) + gB\frac{k^2}{m^2}\right] = 0 \tag{54}$$

It can readily be seen that setting $n_5 = 0$ eliminates the inertial–gravity wave solution. Use of the rotational or geostrophic wind clearly achieves this.

4.5 Filtered Rossby wave (previously the $\sigma = 0$ solution)

Using the previous filtering approximations and now retaining the variation of Coriolis parameter with latitude leads to the following equation for the pressure perturbation ($(\delta p / \rho_0) \equiv \tilde{p}$):

$$\frac{\partial}{\partial t}\left\{\frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + \frac{f_0^2}{gB}\frac{\partial^2 \tilde{p}}{\partial z^2}\right\} + \beta\frac{\partial \tilde{p}}{\partial y} = 0 \tag{55}$$

Putting $\tilde{p}$ proportional to $\exp[\mathrm{i}(kx + ly + mz - \sigma t)]$ gives the dispersion relation:

$$\sigma = -\frac{\beta k}{\left(k^2 + l^2 + m^2\frac{f_0^2}{gB}\right)}$$

and for a uniform basic zonal current $\overline{U}$ (i.e. replacing σ by $(\sigma - k\overline{U})$):

$$\sigma = k\overline{U} - \frac{\beta k}{\left(k^2 + l^2 + m^2\frac{f_0^2}{gB}\right)} \tag{56}$$

Thus these waves, known as <u>Rossby waves</u>, must propagate <u>westward</u> relative to the mean zonal flow since:

$$c_x = \overline{U} - \frac{\beta}{\left(k^2 + l^2 + m^2\frac{f_0^2}{gB}\right)}$$

Clearly Rossby waves are dispersive and long zonal and meridional wavelengths will propagate fastest. However typical values show that large-scale Rossby waves move quite slowly (10 m s^{-1}). From (56) we can deduce that short zonal wavelengths have a group velocity opposite in direction to their phase velocity.

Equation (55) is a form of the barotropic vorticity equation which states that the vertical component of absolute vorticity is conserved following the horizontal motion. A useful physical appreciation of the westward propagation is shown in Fig. 1 (from *Holton*).

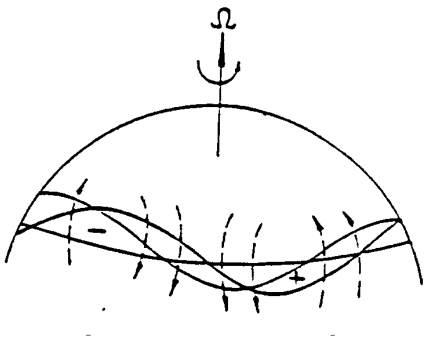

Figure 1. Perturbation vorticity field and induced velocity field (dashed arrows) for a meridionally displaced chain of fluid parcels. The heavy wavy line shows the original perturbation position, and the light line shows the westward displacement of the pattern due to advection by the induced velocity field.

By considering a chain of fluid parcels along a latitude circle and requiring that $\zeta + f$ is conserved then, if $\zeta = 0$ initially, a meridional displacement δy results in:

$$f_0 = f_1 + \zeta_1 \quad \text{(conservation)}$$

but

$$\zeta_1 = f_0 - f_1 = -\beta\delta y \quad \text{(definition)}$$

A sinusoidal displacement therefore gives positive (cyclonic) vorticity for southward displacements and negative (anticyclonic) vorticity for northward displacements. It is then clear from the diagram that this induced flow field advects the chain of fluid parcels such that the wave pattern propagates westwards.

5. SURFACE GRAVITY WAVES

So far we have considered periodic oscillations in unbounded fluids, and the question of appropriate boundary conditions has not arisen. We now examine waves in a region bounded below by a smooth, horizontal, rigid boundary and above by a free surface. A “free” surface as the name implies means that the surface shape responds to the motion within the fluid and cannot be determined ‘a priori' i.e. the fluid motion and the boundary shape must be determined simultaneously for a complete solution of a free-boundary problem.

Any smooth surface, either free or rigid, must be a material boundary. That is, particles adjoining the surface follow the surface contours. It follows that $\mathrm{D}(z-h)/\mathrm{D}t = 0$ at $z = h$, where h is the surface profile, defines this (kinematic) condition. So $w = \mathrm{D}h/\mathrm{D}t$, a condition with reduces to $w = 0$ on a horizontal, rigid boundary. However, this kinematic condition alone is not sufficient to determine the shape of a free surface, and an additional (dynamical) boundary condition must be specified to determine a solution. This extra boundary condition is continuity of pressure. In practice this condition has to be re-expressed in terms of kinematic quantities by using either the momentum or Bernoulli equations.

We will consider the simplest example of surface gravity waves and isolate their physical origin. For this purpose we suppose the fluid to be unstratified $(B = 0)$. Acoustic waves can be eliminated for reasons given earlier, but this has to be done rather carefully so we retain n_2 at present.

With these assumptions (32) reduces to:

$$\frac{d^2}{dz^2}\hat{w} - \frac{n_1}{H_0}\frac{d}{dz}\hat{w} - n_4\sigma^2\left(\frac{k^2}{\sigma^2 - f^2} - \frac{n_2}{c^2}\right)\hat{w} = 0 \tag{57}$$

For simplicity we will look at the behaviour of small amplitude waves in a fluid with a free surface $h(x, t) = A\cos(kx + \sigma t)$ above which the pressure is assumed constant (e.g. air–sea interface, model of waves on inversion). The lower boundary is assumed to be a rigid horizontal surface at $z = 0$.

With the above comments in mind the (non linear) boundary conditions are:

(i) $w = 0$ at $z = 0$

(ii) $w = \mathrm{D}h/\mathrm{D}t$ at $z = h$

(iii) pressure continuous at the free surface, which in this example is equivalent to $\mathrm{D}p/\mathrm{D}t = 0$ at $z = h$.

The linearized forms of these conditions are:

(i) $\delta w = 0$ at $z = 0$

(ii) $\delta w = \partial h / \partial t$ at $z = h$

(iii) $\partial(\delta p/\rho_0)/\partial t = g\delta w$ at $z = h$ which, by using (29) and the condition that $B + g/c^2 = 1/H_0$, can be shown to be equivalent to

$$\frac{d}{dz}\hat{w} - \left[\frac{(n_1 - n_2)}{H_0} + g\frac{k^2}{\sigma^2 - f^2}\right]\hat{w} = 0 \quad \text{at} \quad z = h \tag{58}$$

From (58) we see that if we eliminate acoustic waves by setting $n_2 = n_3 = 0$, we must also set $n_1 = 0$ in the boundary condition otherwise an oscillation, spurious in the sense that it cannot occur in the exact equations, will arise through an inconsistent approximation.

Equation (57), with the above boundary conditions, can readily be solved for all wavelengths but it is instructive to consider long and short waves independently.

5.1 Long waves

If waves are long to the extent that $kH_0 \ll 1$ and $kh \ll 1$, then the non-hydrostatic term containing the term n_4 in (57) can be neglected by comparison with $(1/H_0)(d\hat{w}/dz)$ and the solution is seen to be:

$$\delta w = \sigma A \frac{\left(1 - \mathrm{e}^{n_1 \frac{z}{H_0}}\right)}{\left(\mathrm{e}^{n_1 \frac{h}{H_0}} - 1\right)} \sin(kx + \sigma t) \tag{59}$$

where $\sigma^2 = f^2 + gH_0k^2\left(1 - \mathrm{e}^{-n_1\frac{h}{H_0}}\right)$.

If the height scale is negligible by comparison with the density scale height left($h \ll H_0$)

$$\frac{\sigma}{k} = \sqrt{\left(\frac{f^2}{k^2} + gh\right)} + \mathrm{O}\left(\frac{h}{H_0}\right)$$

and is, therefore, the propagation speed of long surface waves in a liquid or an atmospheric layer which is much shallower than 7 km. For instance, long waves on the boundary-layer inversion, $h \approx 1$ km and so $c \approx 100$ m s^{-1}.

If, on the other hand, $h \gg H_0$ (for example in a deep gas) then $\frac{\sigma}{k} = \sqrt{\left(\frac{f^2}{k^2} + gH_0\right)}$.

It is easy to show that the earths rotation only becomes important for synoptic scales or longer. For the earths atmosphere these deep long waves propagate at about 270 m s^{-1}. These long waves are often referred to as "shallow water" waves and the equation set resulting from (17)–(21) with the relevant approximations (normally with $\partial/\partial y \neq 0$) form the "shallow water equations". These provide a popular equation set for analytical/dynamical studies and also for testing and designing numerical integration schemes.

5.2 Short waves

When $kH_0 \gg 1$ and $kh \gg 1$ the term involving n_4 cannot be neglected whereas rotation can obviously be so. The solution can be shown to be:

$$\delta w = -\sigma A \frac{\sinh(kz)}{\sin(kh)} \exp\left[n_1 \frac{(z-h)}{2H_0}\right] \sin(kx + \sigma t) \tag{60}$$

where $\sigma^2 = gk\tanh(kh)$. Hence for $kh \gg 1$, $\sigma/k \approx \sqrt{g/k}$, a much slower phase speed than the long waves.

If desired, free-surface waves can be eliminated by imposing rigid upper and lower boundary conditions.

6. EQUATORIAL WAVES

Near the Equator atmospheric waves acquire a rather different character and what were clearly distinct mechanisms cease to be so. Following *Matsuno* (1966) we will study equatorial waves using the shallow water equations with $f = \beta y$, known as the equatorial beta-plane, and including $\partial/\partial y \neq 0$, and linearize again about a state of no motion. Hence:

$$\begin{aligned} \frac{\partial u'}{\partial t} - \beta y v' + g\frac{\partial h'}{\partial x} &= 0 \\ \frac{\partial v'}{\partial t} + \beta y u' + g\frac{\partial h'}{\partial y} &= 0 \\ \frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) &= 0 \end{aligned} \tag{61}$$

EXERCISE:

Derive the above equations from (1)–(5).

Assuming wavelike forms for $(u', v', h') = (\hat{u}, \hat{v}, \hat{h})\exp[i(kx + \sigma t)]$ in the east–west direction, but retaining the y-variation explicitly leads to the differential equation:

$$\frac{\mathrm{d}^2\hat{v}}{\mathrm{d}y^2} + \left(\frac{\sigma^2}{gH} - k^2 + \frac{k\beta}{\sigma} - \frac{\beta^2}{gH}y^2\right)\hat{v} = 0 \tag{62}$$

It is convenient to non-dimensionalize the equation using:

$$y^2 = \frac{\sqrt{gH}}{\beta}\lambda^2;\; k^2 = \frac{\beta}{\sqrt{gH}}\mu^2;\; \sigma^2 = \sqrt{gH}\beta\omega^2$$

(62) then becomes:

$$\frac{d^2\hat{v}}{d\lambda^2} + \left(\omega^2 - \mu^2 + \frac{\mu}{\omega} - \lambda^2\right)\hat{v} = 0 \tag{63}$$

This equation is a form of Schrödinger equation with solutions, if

$$\omega^2 - \mu^2 + \frac{\mu}{\omega} = 2n + 1 \qquad (n = 0, 1, 2, \ldots), \tag{64}$$

of the form $\hat{v} = v_0\exp(-\lambda^2/2)\mathcal{H}_n(\lambda)$,

where $\mathcal{H}_n$ is an nth order Hermite polynomial ($\mathcal{H}_0 = 1,\; \mathcal{H}_1 = 2\lambda,\; \mathcal{H}_2 = 2(2\lambda^2 - 1)$).

Clearly the solutions are 'trapped' near the Equator by the Gaussian function $\exp(-\lambda^2/2)$.

The dispersion or frequency relationship (64) is cubic in ω:

$$\omega^3 - (\mu^2 + 2n + 1)\omega + \mu = 0 \tag{65}$$

The three distinct roots represent a pair of inertial gravity waves and one Rossby wave. The solutions can conveniently be studied by considering the cases (a) $\omega^2 \sim \mu^2$ and (b) $|\omega| \ll \mu$ (k large), hence:

$$\omega_{1,2} = \mp\sqrt{\mu^2 + 2n + 1} \tag{66}$$

$$\omega_3 = \frac{\mu}{\mu^2 + 2n + 1} \tag{67}$$

Returning to dimensional variables, we obtain for the phase velocities (defined by $c \equiv -\sigma/k$):

$$c_{1,2} = \mp\sqrt{gH}\sqrt{1 + \frac{\beta(2n+1)}{k^2\sqrt{gH}}}$$
$$c_3 = -\frac{\beta}{\left(k^2 + \frac{\beta(2n+1)}{\sqrt{gH}}\right)},$$

Obviously $c_{1,2}$ are a pair of eastward and westward propagating inertial gravity waves while c_3 is a westward propagating Rossby wave.

For $n \geq 1$ these three solutions are distinct but for $n = 0$, (65) can be factorized:

$$(\omega - \mu)(\omega^2 + \omega\mu - 1) = 0$$

The root $\omega = \mu$ is unacceptable (show) but the root $\omega_1 = -\mu/2 - \sqrt{(\mu/2)^2 + 1}$ is an eastward propagating inertial gravity wave and the root omega $\omega_2 = -\mu/2 + \sqrt{(\mu/2)^2 + 1}$ is called a mixed Rossby gravity wave; this is because, as $\mu \to 0$, $\omega_2 \to 1$ (the limiting case of gravity waves (see (66)), while, as $\mu \to \infty$, $\omega_2 \to 0$, the limiting case of Rossby waves (see (67)).

One other important special case is when $n = -1$. This corresponds to a wave with $\hat{v}$ identically zero. This has solutions (see (61)) $c = -\sigma/k = \pm\sqrt{gH}$, where only the positive root is acceptable (the negative root violates the equatorial beta-plane assumption. See also (65) with $n = -1$). This fast-moving eastward propagating wave is the Kelvin wave (see Fig. 2). These waves occur commonly in the ocean as waves along coastlines (decaying exponentially away from the coast).

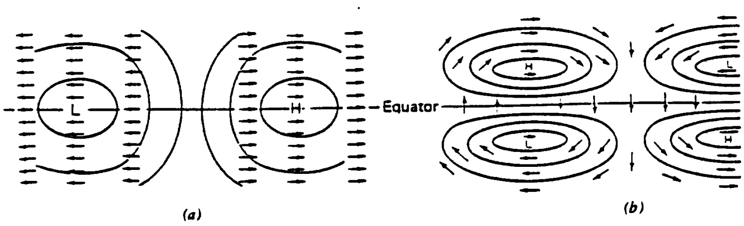

Figure 2. Velocity and pressure distributions in the horizontal plane for (a) Kelvin waves, and (b) Rossby–gravity waves (from Matsuno “Quasi-geostrophic motions in the equatorial area”, J. Meteorol. Soc. Japan, 1966).

Data studies of the equatorial stratosphere have identified both very long wavelength Kelvin and mixed Rossby–gravity waves. Considerable recent interest has been shown in these tropical waves in studies of diabatic forcing in the tropics and its impact on higher latitudes (*Gill*, 1980 etc.). Fig. 3 summarises these equatorial wave characteristics.

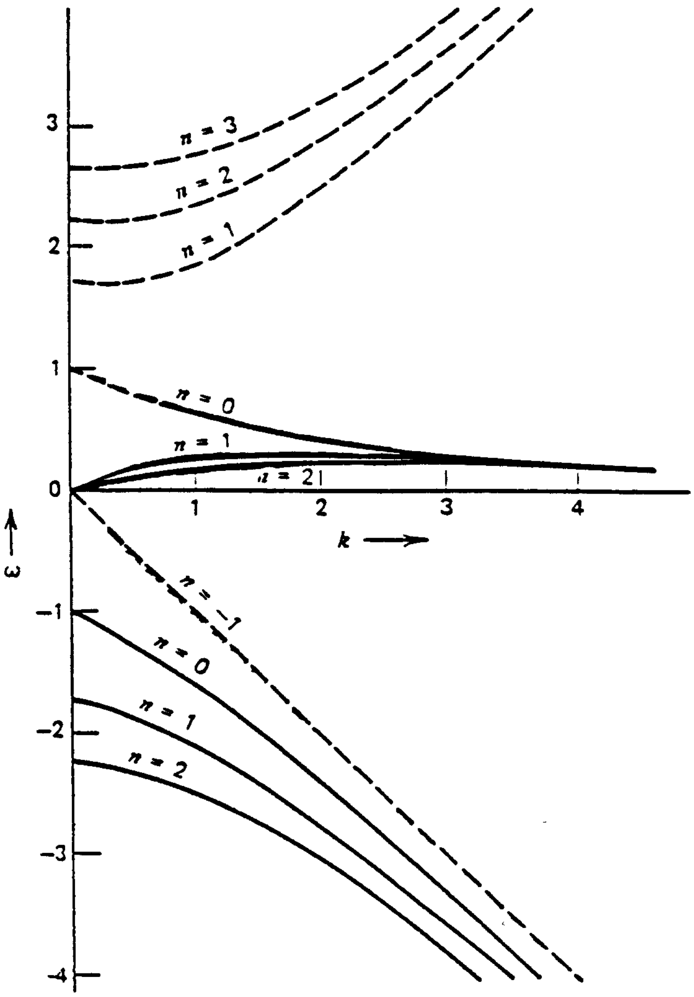

Figure 3. Non-dimensional frequencies from (65) as a function of wavenumber. The lines indicate the following types of waves: Eastward propagating inertial–gravity wave (thin solid line); westward propagating inertial–gravity wave (thin dashed line); Rossby wave (thick solid line); Kelvin wave (thick dashed line). (After *Matsuno* 1966)

7. SUMMARY DIAGRAM AND MODELLING IMPLICATIONS

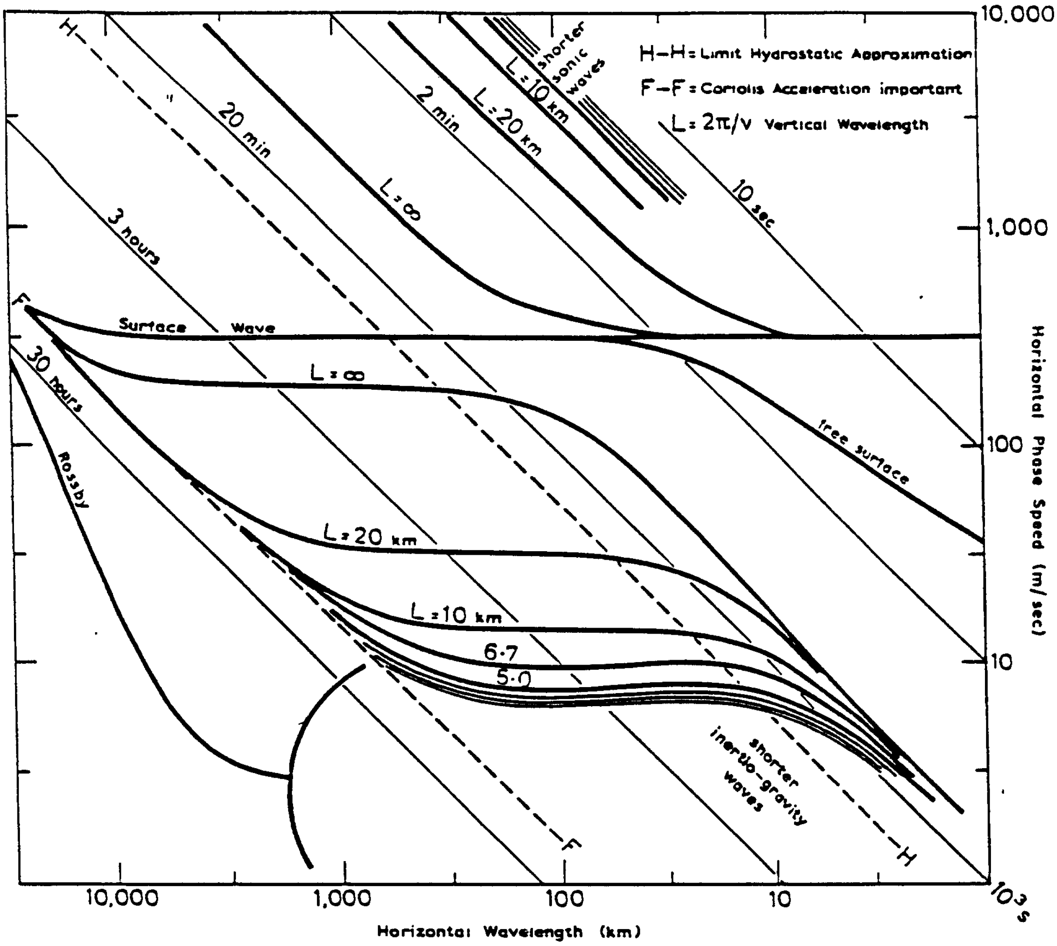

Figure 4. Waves in a compressible atmosphere (from J. S. A. Green — Dynamics lecture notes)

Fig. 4 summarises the basic wave types. It is a dispersion diagram of horizontal phase speed plotted against horizontal wavelength (both logarithmic scales) so isopleths of wave period are straight lines of unit slope with intercept $\log\sigma$. Some points of interest are:

(i) The short vertical wavelength acoustic waves which, while having phase speeds normal to their wave fronts equal to $c = \sqrt{\gamma RT}$, have very large horizontal phase speeds (with wave fronts almost parallel to the ground).

(ii) Mesoscale gravity waves are almost non-dispersive.

(iii) Rossby waves for long wavelengths vary like k^{-2} while the shorter wavelengths include the mixed Rossby-gravity mode.

In previous sections we have identified several wave types with a wide range of frequencies. The choice of time step in a numerical model is dominated by the highest frequency waves (see Lecture Note 1.4). Thus for example, acoustic waves have a frequency $\sigma = c\sqrt{k^2 + m^2}$ and for a typical large-scale model $k/m = \Delta z/\Delta x \sim 10^{-2}$. Hence σ_{MAX} is dominated by the vertically propagating acoustic waves and it is desirable to filter out these vertically propagating waves by assuming hydrostatic balance.

Horizontally propagating acoustic waves (Lamb) pose less of a problem since their frequencies are not much higher than the long gravity waves discussed earlier left ($(\sigma_a)/\sigma_g \sim \sqrt{\gamma RT/gH} \geq 1$). Nevertheless various numerical techniques have been devised to ease this restriction (see Lecture Note 1.4).

We could make the anelastic approximation ($\partial\rho/\partial t = 0$ in the continuity equation) but retain the non-hydrostatic terms. An equation set based on (17)–(21) with $n_2 = n_3 = 0$ then allows us to predict u, v and w; however these components must satisfy the continuity equation. Differentiating (20) with respect to time and substituting from (17), (18) and (19) gives a diagnostic Poisson-type equation for the pressure of the form $\nabla^2(\delta p/\rho_0) = F(u, v, w, \ldots)$ which must be inverted each time step. This is a relatively expensive procedure and not justifiable for large-scale models because the hydrostatic approximation is well-satisfied. For much smaller scale models where $\Delta x \geq \Delta z$ the decision is no longer straightforward and generally either the anelastic approximation is made or sound waves are retained and treated by some sophisticated numerical technique.

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