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Large-amplitude nonlinear stability results for atmospheric circulations

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Abstract

It is shown how large-amplitude stability results for flows governed by potential vorticity conservation can be obtained by geometric arguments using rearrangements of functions. The resulting stability bounds are more general than those found by weakly nonlinear methods, as they allow for non-smooth solutions. The method also gives a framework for the rigorous treatment of the effects of mixing by increasingly fine-scale filamentation. The methods are applied to the semi-geostrophic shear flow problem where the results can be compared with other methods. It is shown that under the definitions used, there are no non-zonal nonlinearly stable states in this problem. In the baroclinic case it is shown how potential vorticity conservation reduces the 'available energy' for the transient flow.

1 Introduction

In this paper we discuss the dynamics of large-scale atmospheric circulations using theories based on Hamiltonian mechanics. This is appropriate if we assume that the effect of physical forcing is to create air masses with different properties, but that the rate of change of airmass properties is slow compared with the time-scale of potential vorticity advection. We then regard the internal dynamics as energy conserving, with potential temperature and potential vorticity conserved following fluid particles. The minimum energy state that is consistent with potential temperature conservation is a 'rest' state, where the air masses are rearranged so that the potential temperature surfaces are horizontal. It is well known that the energy available to the internal dynamics is the difference between the actual energy and the energy of the 'rest' state. In this paper we explore the additional restriction that the available energy is only the excess over that of a minimum energy state obtained by rearranging both potential vorticity and potential temperature.

There is an important difference between minimising the energy with respect to rearrangements of potential temperature, and minimising the energy with respect to simultaneous rearrangements of potential vorticity and potential temperature. It can be proved that the minimum energy state with respect to potential temperature conservation is a state with potential temperature monotonically increasing with height. States which could be reached by mixing the fluid have a higher energy than the monotonic state. However, in most cases, we show that the minimum energy consistent with potential vorticity conservation is reached by increasingly fine-scale filamentation, and is actually the energy of a mixed state of the fluid. This issue is discussed by Burton and Nycander (1999), henceforward referred to as BN, in the context of quasi-geostrophic theory. Much of this paper is therefore concerned with identifying which states can be reached by mixing.

The advantage of using rearrangements is that stability can be established with respect to all displacements, whether smooth or not. The method is thus more general than the energy-Casimir method used, for instance, by Kushner and Shepherd (1995ab). It is also possible to derive results by geometrical arguments which may be difficult to establish by algebraic methods. For instance, Ren (2000a) shows that it is very difficult to apply Arnold's stability methods to semi-geostrophic theory because of the nonlinear form of the potential vorticity. This form of potential vorticity has a natural geometric interpretation, and can be readily used with rearrangement methods.

These methods are applicable with any 'balanced model' that conserves energy and has a conserved potential vorticity. We use the semi-geostrophic model in this paper. BN use quasi-geostrophic theory. The methods of analysis required are quite different for the two sets of equations, and the conclusions might well differ in interesting ways. We obtain stability results similar to those of Kushner and Shepherd (1995ab) and Ren (2000ab) for shear flows using Kelvin's principles, (Thomson, 1910): firstly that steady states are stationary points of the energy under rearrangements of the potential vorticity and potential temperature, and secondly that stable steady states are extrema of the energy under these rearrangements. Under these definitions, we show that the extremising states are zonally symmetric, and therefore there are no non-zonal nonlinearly stable states.

We also show that the minimum energy state for a baroclinic flow will in general have non-zero kinetic energy, so that the available energy for the transient flow will be less than the available energy calculated by subtracting a rest state. This minimum energy state will have vertical variations in its potential vorticity structure unless the given potential vorticity variations have a horizontal scale much larger than the Rossby radius of deformation. In that case the potential vorticity structure of the minimising state becomes vertically uniform.

2 Basic method and generic results

2.1 Generic equations

We assume initially that the exact internal dynamics are governed by the incompressible Euler equations on a region Ω which is doubly periodic in (x,y) and confined by rigid boundaries at $z=0,H$ in the vertical:

$$\begin{aligned} \frac{D\mathbf{v}_h}{Dt} + (-fv, fu) + \nabla_h\phi &= 0 \\ \frac{Dw}{Dt} - g\theta/\theta_0 + \frac{\partial\phi}{\partial z} &= 0 \\ \nabla.\mathbf{v} &= 0 \\ \frac{D\theta}{Dt} &= 0 \\ w &= 0 \text{ on } z=0,H \end{aligned} \tag{1}$$

Here, $\mathbf{v}=(u,v,w)$ is the wind vector with $\mathbf{v}_h=(u,v)$ its horizontal part. ϕ is the geopotential, θ is the potential temperature and θ_0 is a reference value of θ. f is the Coriolis parameter assumed constant in this geometry. Application of these to the atmosphere requires a form of Boussinesq approximation (McWilliams (1985)). Equations (1) conserve an energy integral, and also conserve potential temperature and the Ertel potential vorticity

$$q = (f+\zeta).\nabla\theta \tag{2}$$

following fluid particles. However, potential vorticity conservation alone does not give complete information about the velocity field. We can only determine the complete flow from knowledge of the potential vorticity if we use an approximate set of equations, appropriate to large scale atmospheric flow. A generic form of such equations takes the form, Hoskins et al. (1985):

$$\begin{aligned} \frac{DQ}{Dt} &= 0 \\ \frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + \mathbf{v}.\nabla \\ \mathcal{H}(\mathbf{v},\theta) &= Q \end{aligned} \tag{3}$$

Q is the potential vorticity, which may be the Ertel potential vorticity q defined by (2) or an approximation to it. Inverting the operator $\mathcal{H}$ determines $\mathbf{v}$ and θ given Q. Usually this requires $\mathcal{H}$ to be elliptic and boundary conditions to be specified (and there may be additional technical restrictions). Particular care is needed in spherical geometry, where the potential vorticity goes to zero at the equator. This case is not treated in this paper. The simplest example is to seek fields which are in geostrophic and hydrostatic balance, and have a specified Ertel potential vorticity. If, in addition, only the vertical component of the vorticity is considered, this yields the equation

$$\left(\frac{\partial}{\partial x}(f^{-1}\frac{\partial\phi}{\partial x})+\frac{\partial}{\partial y}(f^{-1}\frac{\partial\phi}{\partial y})+f\right)\frac{\theta_0}{g}\frac{\partial^2\phi}{\partial z^2}=q \tag{4}$$
$$\frac{\partial\phi}{\partial z}\text{ given on } z=0,H$$

This can be solved for the geopotential, and hence the geostrophic wind and temperature fields. However, it gives no information about the time evolution. Therefore in practice the inversion has to be carried out assuming a balanced approximation to (1) that goes beyond geostrophy. Many balanced models can be written in the generic form (3), such as quasi-geostrophic theory, semi-geostrophic theory, and some forms of the nonlinear balance equations. The boundary conditions are different in each case. Systems of equations derived this way will imply the equations

$$\begin{aligned}\nabla.\mathbf{v} &= 0 \\ \frac{D\theta}{Dt} &= 0 \\ w &= 0 \text{ on } z=0,H\end{aligned} \tag{5}$$

from (1), but will correspond to an approximation of the momentum equations in (1). Equations (3) and (5) state that the potential vorticity Q and potential temperature θ are simultaneously rearranged by the incompressible velocity field $\mathbf{v}$. This means that Q and θ stay bounded by their initial values, and the volume of fluid with given values of Q and θ is conserved. However, the values may be mixed on increasingly finer scales as we illustrate in the next subsection.

Kelvin's principle (Thomson, 1910) applied to equations (3) and (5) states that steady states are stationary points of the energy with respect to rearrangements of the potential vorticity and potential temperature. Stable steady states correspond to maxima or minima of the energy under such rearrangements. The variational calculation of BN identifies a maximum energy state. Thus it is likely that there will be only a small number of globally stable steady states corresponding to the maximum and the minimum energy that are obtainable over the whole class of rearrangements, including mixing, but there may also be a large class of locally stable states which are extrema of the energy subject to physically reasonable displacements.

2.2 Properties of rearrangements

We need to make extensive use of the concept of *rearrangement* of a function. We begin with an intuitive definition. Suppose f is a function defined on a bounded region of real space, thought of as a continuum of fluid particles. Suppose we exchange the particle positions, with each particle retaining its value of f. This yields a function g, which is a *rearrangement* of f. Roughly speaking, for any given collection of values, the set of points where f takes those values has the same size as the corresponding set for g. We review some mathematical definitions and properties of rearrangements which we need to make our ideas precise. This material is largely taken from Douglas (2001), henceforward referred to as D, which should be referred

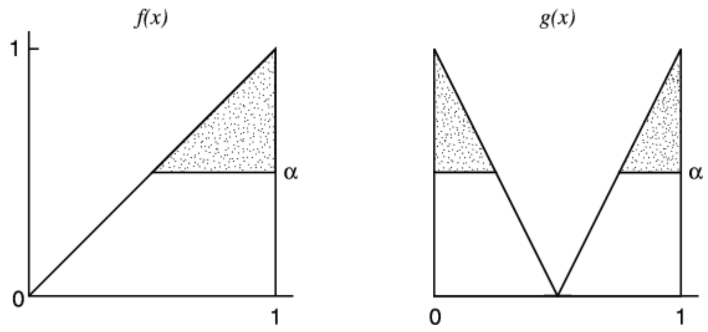

Figure 1: Graphs of $f(x)$ and a rearrangement $g(x)$ of $f(x)$. The shaded regions are those bounded below by the line $y = \alpha$ and above by the graphs of f and g; the areas are the same for every choice of α.

to for further detail. If $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ are two non–negative integrable functions, i.e $\int_\Omega f(\mathbf{x})d\mathbf{x} < \infty$, $\int_\Omega g(\mathbf{x})d\mathbf{x} < \infty$, then g is said to be a *rearrangement* of f if

$$\int_\Omega (f(\mathbf{x}) - \alpha)_+ d\mathbf{x} = \int_\Omega (g(\mathbf{x}) - \alpha)_+ d\mathbf{x} \tag{6}$$

for each $\alpha > 0$ and where the $+$ subscript denotes taking the positive part of the function. This definition is illustrated in Fig.1. We write the set of all functions which are a rearrangement of a given f as $\mathcal{R}(f)$.

We will be considering the problem of finding the maximum or minimum of the energy which can be obtained by rearranging a given potential vorticity distribution on isentropic surfaces. A classical approach would be to choose a maximising (or minimising) sequence, and extract a subsequence converging to a maximiser (or a minimiser). However, this limit might not be a rearrangement; there is the possibility of 'mixing'. It is possible to construct an increasingly fine-grained sequence of rearrangements, which converge in a weak sense to a smoothed potential vorticity distribution which is not a rearrangement. We give a simple one-dimensional example. Let the function f_0 on $[0, 1]$ be defined by

$$f_0(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2], \\ 1 & \text{if } x \in (1/2, 1]. \end{cases} \tag{7}$$

Define, for $n \in \mathbb{N}$,

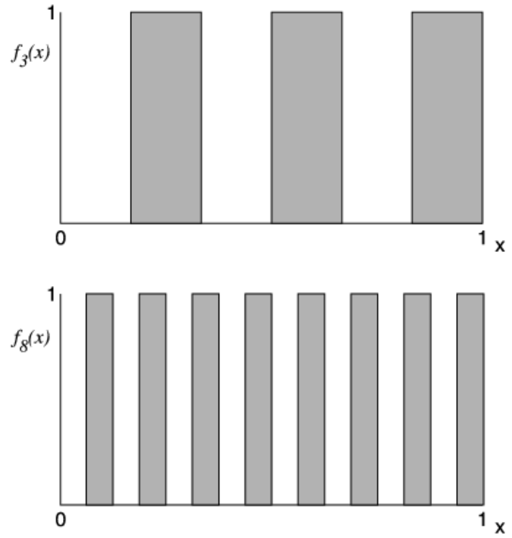

Figure 2: Graphs of $f_3(x)$ and $f_8(x)$ as defined in (8).

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0 & \text{if } x \in (m/n, (2m+1)/2n], \\ 1 & \text{if } x \in ((2m+1)/2n, (m+1)/n]. \end{cases} \tag{8}$$

where $m = 0, 1, \ldots, n-1$. The functions f_3 and f_8 are illustrated in Figure 2. For each $n \in I\!N$, f_n is equal to zero on a set of length $1/2$, and equal to 1 on a set of length $1/2$, therefore f_n is a rearrangement of f_0 for each $n \in I\!N$. However, given any square integrable function g on $[0,1]$ (i.e. $\int_0^1 g^2(x)dx < \infty$), it may be shown that $\int_0^1 f_n g dx \to 1/2 \int_0^1 g dx$ as $n \to \infty$, that is f_n converges weakly to the constant function with value $1/2$, which is not a rearrangement of f_0.

This occurs in situations like the filamentation at the stratospheric vortex edge. Physically, including these limit functions can be thought of as allowing for a small but finite viscosity or conductivity. If we adopt the strategy of trying to extract weakly convergent subsequences from an energy extremising sequence, we need to include these limits, as discussed by D, section 2.4, and BN. (In the problem solved by BN, additional work showed there is a solution which is a rearrangement.) A set is said to be *weakly sequentially compact* if for any sequence composed of elements of the set, we can find a subsequence which converges weakly to an element of the set. For a given f, we seek the smallest such set which contains $\mathcal{R}(f)$. It may be characterised (see for example Ryff 1970) as the closed convex hull of the set $\mathcal{R}(f)$, the intersection of all the (strongly) closed convex sets that contain $\mathcal{R}(f)$; we denote this set $\mathcal{C}(f)$. Douglas (1994) showed that $g \in \mathcal{C}(f)$ implies $(\int_\Omega g(\mathbf{x})^p d\mathbf{x})^{1/p} \leq (\int_\Omega f(\mathbf{x})^p d\mathbf{x})^{1/p}$ for every $p > 1$. This is a formal expression of the lack of robustness of the higher order moments of the potential vorticity as constants of the motion, and that a state which extremises the energy may be obtained by selective decay of the higher order moments. These issues are discussed, for instance, by Robert and Sommeria (1991), and Larichev and McWilliams (1991).

We will make use of the following characterisation of $\mathcal{C}(f)$ by Douglas (1994):

$$\mathcal{C}(f) = \{g \geq 0 : \int_{\Omega} (g(\mathbf{x}) - \alpha)_+ d\mathbf{x} \leq \int_{\Omega} (f(\mathbf{x}) - \alpha)_+ d\mathbf{x} \\ \text{for each } \alpha > 0, \int_{\Omega} g(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) d\mathbf{x}.\}, \tag{9}$$

where the $+$ subscript is defined as before. This enables us to prove that rearranging a function f, followed by taking a local average, gives a member of $\mathcal{C}(f)$. We make extensive use of this result in the rest of the paper.

Theorem 1 Let non-negative $f : \Omega \to \mathbb{R}$ be square integrable, and suppose $g \in \mathcal{R}(f)$. For a set $\Gamma \subset \Omega$ of positive volume $\mu(\Gamma)$, define

$$h(x) = \begin{cases} \frac{1}{\mu(\Gamma)} \int_{\Gamma} g(\mathbf{x}) d\mathbf{x} \equiv \bar{g} & \text{if } x \in \Gamma, \\ g(x) & \text{if } x \in \Omega \backslash \Gamma. \end{cases}$$

Then $h \in \mathcal{C}(f)$.

Proof: We first prove the intermediate result that

$$\int_{\Gamma} (\bar{g} - \alpha)_+ d\mathbf{x} \leq \int_{\Gamma} (g(\mathbf{x}) - \alpha)_+ d\mathbf{x} \tag{10}$$

for every $\alpha > 0$. If $\alpha \geq \bar{g}$, (10) follows trivially as the left hand integral is zero. Otherwise $0 < \alpha < \bar{g}$. Define $\Gamma_1 = \{\mathbf{x} \in \Gamma : g(\mathbf{x}) > \alpha\}, \Gamma_2 = \{\mathbf{x} \in \Gamma : g(\mathbf{x}) \leq \alpha\}$, and $\mu(\Gamma_1), \mu(\Gamma_2)$ to be their respective volumes. Then

$$\begin{aligned} \int_{\Gamma} (\bar{g} - \alpha)_+ d\mathbf{x} &= \mu(\Gamma)(\bar{g} - \alpha) = \int_{\Gamma} g(\mathbf{x}) d\mathbf{x} - \alpha\mu(\Gamma) \\ &= \int_{\Gamma_1} g(\mathbf{x}) d\mathbf{x} + \int_{\Gamma_2} g(\mathbf{x}) d\mathbf{x} - \alpha\mu(\Gamma_1) - \alpha\mu(\Gamma_2) \\ &= \int_{\Gamma} (g(\mathbf{x}) - \alpha)_+ d\mathbf{x} + \int_{\Gamma_2} g(\mathbf{x}) d\mathbf{x} - \alpha\mu(\Gamma_2) \leq \int_{\Gamma} (g(\mathbf{x}) - \alpha)_+ d\mathbf{x}. \end{aligned}$$

Now for $\alpha > 0$,

$$\begin{aligned} \int_{\Omega} (h(\mathbf{x}) - \alpha)_+ d\mathbf{x} &= \int_{\Gamma} (h(\mathbf{x}) - \alpha)_+ d\mathbf{x} + \int_{\Omega \backslash \Gamma} (h(\mathbf{x}) - \alpha)_+ d\mathbf{x} \\ &= \int_{\Gamma} (\bar{g} - \alpha)_+ d\mathbf{x} + \int_{\Omega \backslash \Gamma} (g(\mathbf{x}) - \alpha)_+ d\mathbf{x} \\ &\leq \int_{\Gamma} (g(\mathbf{x}) - \alpha)_+ d\mathbf{x} = \int_{\Gamma} (f(\mathbf{x}) - \alpha)_+ d\mathbf{x}. \end{aligned} \tag{11}$$

By way of explanation, the inequality in (11) follows by (10), and the equality because $g \in \mathcal{R}(f)$.

Finally

$$\int_{\Omega} h(\mathbf{x}) d\mathbf{x} = \int_{\Gamma} \bar{g} d\mathbf{x} + \int_{\Omega \backslash \Gamma} g(\mathbf{x}) d\mathbf{x} = \int_{\Omega} g(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) d\mathbf{x},$$

noting $g \in \mathcal{R}(f)$. The result follows from (9).

Remark In the one-dimensional example above, with $f = f_0$ as in (7), it can be shown (by using the characterisation (9)) that any integrable function g on $[0,1]$ satisfying $0 \leq g(x) \leq 1$ for each $x \in [0,1]$, and $\int_0^1 g(x) dx = 1/2$, belongs to $\mathcal{C}(f_0)$. This illustrates that $\mathcal{C}(f_0)$ may be a large class of functions, in particular it includes the constant value $1/2$ as illustrated earlier.

A particularly useful result in our context is that $\mathcal{R}(f)$ is weakly dense in $\mathcal{C}(f)$, that is for every $h \in \mathcal{C}(f)$ we can find a sequence $(f_n) \subset \mathcal{R}(f)$ which converges weakly to h.

To proceed further, we consider the one-dimensional case and thus take f to be a non-negative function on $[0,1]$. It is shown in D that there is an (essentially) unique rearrangement of f which is an increasing function. Write is as as $\tilde{f}$. We will need to make use of the following result which follows from standard rearrangement inequalities:

Theorem 2 If $f(x)$ is non-negative and monotonically increasing for $x \in [0,1]$, and $g(x)$ is non-negative and bounded, then $\int_0^1 f(x)h(x)dx$ is maximised for $h \in \mathcal{C}(g)$ by the monotonically increasing rearrangement of g, and is minimised by the monotonically decreasing rearrangement of g. The reverse statements apply if $f(x)$ is non-negative and monotonically decreasing.

Proof D, section 2.2, quotes the standard inequality:

$$\int_0^1 f(x)h(x)dx \leq \int_0^1 \tilde{f}(x)\tilde{h}(x)dx \tag{12}$$

where $\tilde{f}, \tilde{h}$ are the increasing rearrangements of f, h respectively. We have $f(x) = \tilde{f}(x)$ by assumption. Consider a sequence of members h_n of $\mathcal{R}(g)$. It is sufficient to do this because $\mathcal{R}(g)$ is weakly dense in $\mathcal{C}(g)$. Then $\tilde{h}_n = \tilde{g}$ and in particular

$$\int_0^1 f(x)h_n(x)dx \leq \int_0^1 f(x)\tilde{g}(x)dx \tag{13}$$

If h is the weak limit of h_n, the inequality (13) then also applies to h, by the definition of a weak limit, giving the first result.

Set $G = \text{Max}(g(x), x \in [0,1]$. Then $\hat{g} = G - g$ is non-negative on $[0,1]$. If $h_n \in \mathcal{R}(g)$, then $G - h_n \in \mathcal{R}(G-g)$ and (12) gives

$$\int_0^1 f(x)(G - h_n(x))dx \leq \int_0^1 f(x)(G - \hat{g}(x))dx \tag{14}$$

where $\hat{g}$ is the decreasing rearrangement of g. Cancelling $\int f(x)Gdx$ and changing sign gives

$$\int_0^1 f(x)h_n(x)dx \geq \int_0^1 f(x)\hat{g}(x)dx \tag{15}$$

as required. (12) also applies if $\tilde{f}, \tilde{h}$ are the decreasing rearrangements of f, h. The results for decreasing f then follow by systematically using $\tilde{f}, \tilde{g}, \tilde{h}$ to denote decreasing rearrangements in the arguments above.

2.3 Minimising energy with respect to PV rearrangements, one- dimensional case

If g represents a velocity variable, then the potential vorticity (2) will depend on the derivative g' of g. Suppose f' represents a background potential vorticity associated with the Earth's rotation, and g' the actual potential vorticity. Then the velocity will be the integral of $(g' - f')$, and the energy will take the form

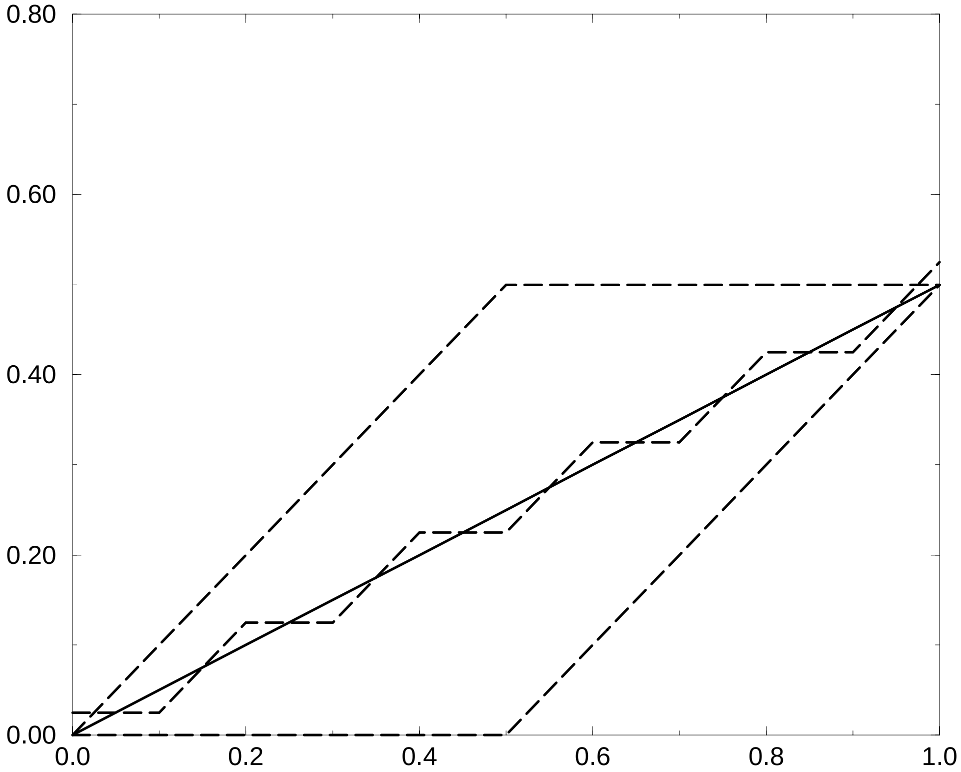

Figure 3: Graphs of $f(x)$ (solid line) and various choices of $h(x)$ obtained by setting $h'(x)$ to be a rearrangement of $g'(x)$. These choices are the monotonically increasing and decreasing rearrangements, and the choice $h'(x) = f_5(x)$ as defined in (8).

$$E = \int_0^1 (g(x) - f(x))^2 dx \tag{16}$$

Typically, there will also be an angular momentum constraint which means that $\int_0^1 g(x)dx$ must be preserved under the variations considered. In some cases, such as the shear flow problem treated in the next section, this is trivial because it determines the constant of integration for calculating g from g'. In other cases, such as the spherical problem, we have an additional condition that $h(0) = g(0)$ is given, and the angular momentum constraint becomes very restrictive. Consider the following generic problem:

Problem 1 Given f and g non-negative and monotonically increasing, define $\mathcal{C}'(g)$ to be the class of functions $h : [0,1] \rightarrow \mathcal{R}$ satisfying $\int_0^1 h(x)dx = \int_0^1 g(x)dx$; $h'(x) \in \mathcal{C}(g'(x))$. Find the minimiser of $E = \int_0^1 (h(x) - f(x))^2 dx$ for $h \in \mathcal{C}'(g)$.

Note that we require g to be monotonically increasing to ensure that g', the potential vorticity, is non-negative. A balance assumption such as (3) normally requires non-negative potential vorticity so that the inversion problem is well-posed.

Take as an example, (Fig. 3), the case where $f(x) = x/2$ and $g'(x)$ is the function $f_0(x)$ defined in (7), so that $g(x) = 0 : 0 < x < \frac{1}{2}; = \frac{1}{2}(x - \frac{1}{2}) : \frac{1}{2} < x < 1$. It is intuitively clear that the largest energy will be obtained by choosing h' to be either the monotonically increasing or decreasing rearrangement of g'. The smallest energy will be obtained by choosing h' to oscillate about $\frac{1}{2}$, the value of f'. If $h'(x) = f_n(x)$, where $f_n(x)$ is as defined in (8), then the energy is $O\left(\frac{1}{n^2}\right)$.

This example shows that, in general, it is harder to find minimisers than maximisers because the minimiser will typically involve mixing while the maximiser will be a strict rearrangement. The physically important case for the global stability problem is usually an energy minimiser, because the global maximiser will often correspond to an unreachable or unphysical state. The local stability problem may well be solved by an energy maximiser, as in the study of BN.

Now consider the nature of $\mathcal{C}'(g)$. Theorem 1 shows that it contains all local averages. Thus in particular, given any x_1, x_2, if $g(x)$ is replaced by its linear interpolant between x_1 and x_2, the result is in $\mathcal{C}'(g)$ because $g'(x)$ has been replaced by its average value over the range (x_1, x_2). In the example shown, the linear interpolant between 0 and 1 gives the function $h(x) = x/2$ which is just $f(x)$. Thus $f(x) \in \mathcal{C}'(g)$ and it is possible to find a sequence $h_n(x)$ in $\mathcal{R}(g)$ with weak limit $h(x) = f(x)$ giving zero energy.

Theorem 3 (i) Given any non-negative $g(x) : x \in (0,1)$; $h(x)$ for each $x \in [0,1]$ is maximised for $h \in \mathcal{C}'(g)$: $h(0) = g(0)$ by choosing h' to be the decreasing rearrangement of g'. Consequently $\int_0^1 h(x)dx$ is also maximised by this choice. $h(x)$ is minimised for each x by choosing h' to be the increasing rearrangement of g'.
ii) Given $h(0) = g(0)$, then if $g'(x)$ is monotonically increasing or decreasing, the only member of $\mathcal{C}'(g)$ satisfying all the conditions of Problem 1 is $g(x)$ itself.

Proof . First note that if $\psi(s) = 1, 0 \leq s < x; \psi(s) = 0, x \leq s \leq 1$, then

$$\int_0^1 \psi(s)h'(s)ds = h(x). \tag{17}$$

The inequality (12) also applies if $\tilde{f}, \tilde{h}$ are decreasing rearrangements. Apply this with $f(s) = \psi(s)$. This gives $h(x) \leq \tilde{h}(x)$, where $\tilde{h}$ is generated by setting $\tilde{h}'$ equal to the decreasing rearrangement of h'. Equality only holds if $\tilde{h}' = h'$ and so $\tilde{h} = h$. This proves the first statement in (i), with a similar argument for the reverse case. The statements about the integral follows immediately. Part (ii) follows from part (i) because any rearrangement of a strictly monotonically increasing h', other than the identity map, will increase $\int_0^1 h(x)dx$, contradicting the requirement $\int_0^1 h(x)dx = \int_0^1 g(x)dx$. This also applies to weak limits of rearrangements because the integrand in (17) is linear in h'. The reverse case is proved similarly.

This result is a form of the Charney Stern theorem, showing that the energy cannot be changed by rearranging a monotone potential vorticity distribution without violating the angular momentum constraint. We can see that it comes from a choice of boundary conditions which allow this to be a real constraint on the flow evolution, rather than something that can be enforced afterwards by choosing a constant of integration. Given a non-monotone potential vorticity profile, the energy can typically be reduced under angular momentum conservation by mixing, so filamentation of the potential vorticity will typically occur in a time integration.

In multi-dimensional problems, it is typical that the extremising states are independent of one or more spatial coordinates. If the multi-dimensional rearrangement problem for such symmetric states can be reduced to a one-dimensional problem, it would be possible to use the results proved above. This can be achieved by a coordinate transformation and leads to a definition of energy of the form

$$E = \int_0^1 (g(x) - f(x))^2 \mu(x)dx \tag{18}$$

where μ is a given function representing an area or volume element associated with the increment dx (so in circular symmetry we take $\mu(x) = x$). Minimise E for $h \in \mathcal{T}'(g)$, the convex hull of the weighted rearrangements of g' defined by $(h : [0,1] \to \mathbb{R}) \in \mathcal{T}'(g)$ if $\int_0^1 h(x)\mu(x)dx = \int_0^1 g(x)\mu(x)dx$, and

$$\int_0^1 (h'(x)-\alpha)_+\mu(x)dx \leq \int_0^1 (g'(x)-\alpha)_+\mu(x)dx \tag{19}$$

for each $\alpha > 0$. If we make the change of variable $dy = \mu(x)dx$, and define new functions F, G by $F'(y) = f'(x), F(0) = f(0)$ and $G'(y) = g'(x), G(0) = g(0)$, then the problem of approximating $f(x)$ by a $h(x) \in \mathcal{I}'(g)$ is identical to that of approximating $F(y)$ by $H(y) \in \mathcal{C}'(G)$.

3 Stability of shear flows

3.1 Semi-geostrophic shear flows

Kushner and Shepherd (1995ab) and Ren (2000ab) have recently examined the finite amplitude stability of shear flows using the semi-geostrophic equations. We show how similar results can be obtained for these equations by seeking maximum and minimum energy states under rearrangements of potential vorticity.

The semi-geostrophic system can be written, following Hoskins (1975), as

$$\begin{aligned}
\frac{D\mathbf{v}_g}{Dt} + (-fv, fu) + \nabla_h\phi &= 0 \\
-g\theta/\theta_0 + \frac{\partial\phi}{\partial z} &= 0 \\
\nabla.\mathbf{v} &= 0 \\
\frac{D\theta}{Dt} &= 0 \\
(fv_g, -fu_g) &= \nabla_h\phi \\
w &= 0 \text{ on } z = 0, H
\end{aligned} \tag{20}$$

The notation is the same as (1) with suffix g denoting geostrophic values. In this section we illustrate stability results where the solution region Ω is a channel of width $2D$ and height H, with periodicity $2L$ in the x direction, with f constant. Define geostrophic and isentropic coordinates

$$\begin{aligned}
X &= x + f^{-1}v_g \\
Y &= y - f^{-1}u_g \\
Z &= g\theta/f^2\theta_0
\end{aligned} \tag{21}$$

The periodicity condition means that X takes values in $[-L, L]$, but Y and Z can take any real values. Then we can rewrite the evolution part of (20) as

$$\begin{aligned}
\frac{D(X,Y)}{Dt} &= (u_g, v_g) \\
\frac{DZ}{Dt} &= 0
\end{aligned} \tag{22}$$

The second component of the first equation implies, on integration over x:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{-L}^{L}\int_{-D}^{D}\int_{0}^{H} Y\,dxdydz &= \int_{-L}^{L}\int_{-D}^{D}\int_{0}^{H} v_g\,dxdydz \\ &= \int_{-L}^{L}\int_{-D}^{D}\int_{0}^{H} f^{-1}\frac{\partial p}{\partial x}dxdydz = 0\end{aligned} \tag{23}$$

This is the expression of the angular momentum constraint. Following Cullen and Purser (1989), the equations can be written as an evolution equation for an inverse potential vorticity ρ in the form

$$\frac{D\rho}{Dt} = 0 \tag{24}$$

where

$$\begin{aligned}\frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y} + W\frac{\partial}{\partial Z} \\ U &= f\left(\frac{\partial R}{\partial Y} - Y\right) \\ V &= f\left(X - \frac{\partial R}{\partial X}\right) \\ W &= 0\end{aligned} \tag{25}$$

R is determined from the Monge Ampere equation

$$\det\left(\frac{\partial^2 R}{\partial(X,Y,Z)}\right) = \rho \tag{26}$$

for $(X,Y,Z) \in \Gamma \equiv ((-L,L)\times(-\infty,\infty)\times(-\infty,\infty))$ with the boundary conditions

$$\begin{aligned}\frac{\partial R}{\partial X}(X,Y,Z) &= \tfrac{\partial R}{\partial X}(X+2L,Y,Z) \\ -D \leq \frac{\partial R}{\partial Y} &\leq D \\ 0 \leq \frac{\partial R}{\partial Z} &\leq H\end{aligned} \tag{27}$$

The physical coordinates obey the relation $\nabla R = (x,y,z)$. It is convenient to define

$$\Psi = R - \frac{1}{2}(X^2+Y^2) \tag{28}$$

Ψ acts as a stream function for the flow defined in (24). Integrating the Monge-Ampere equation over Γ gives the compatibility conditions on ρ

$$\begin{aligned}\rho(X+L,Y) &= \rho(X-L,Y) \\ \int_{-L}^{L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho\,dXdYdZ &= 4LDH\end{aligned} \tag{29}$$

Cullen and Purser show that (26) and (27) can be solved given (29). Their proof was made rigorous in the non-periodic case by Benamou and Brenier (1998). In physical space, the solution has X a monotone function of x, Y of y and Z of z. Thus any sequence of solutions that can be generated by integrating these equations in time must preserve this monotonicity property. There is no constraint on the distribution of ρ in X and Y, so the solutions of (26) and (27) can be any rearrangements of ρ which do not imply that particles change their value of Z. This ensures potential temperature conservation. Analogously, in the quasi-geostrophic case treated by BN, it was shown sufficient to consider rearrangements along horizontal surfaces. This is because there is no vertical advection term in the quasi-geostrophic potential vorticity equation.

We now seek to identify stable steady states for a given initial distribution of inverse potential vorticity $\rho = \sigma(X,Y,Z)$. In order to apply Kelvin's principle we seek a class of perturbations which is dynamically consistent with (24) and (27). We first show that this class can be written as $\mathcal{C}_h(\sigma)$, defined by

$$\rho \in \mathcal{C}_h(\sigma) \text{ if } \begin{cases} \rho(.,Z) \in \mathcal{C}(\sigma(.,Z)) \text{ for almost all } Z \\ \int Y\rho dXdYdZ = \int Y\sigma dXdYdZ \end{cases} \tag{30}$$

The first condition restricts the rearrangements of ρ to the (X,Y) variables only, and includes the weak limits. This is similar to the space of 'stratified' rearrangements used by BN. The additional condition is that the mean Y over the particles cannot be changed. This corresponds to the angular momentum conservation equation (23). We write $\mathcal{R}_h(\sigma)$ for functions ρ satisfying (30) with $\mathcal{R}$ replacing $\mathcal{C}$.

3.2 Steady states

We next demonstrate the characterisation of steady states of the semi-geostrophic system in terms of stationary points of the energy with respect to rearrangements. The energy for the problem (20) can be written using (21) as

$$\begin{aligned} E = & \int_{-L}^{L}\int_{-D}^{D}\int_{0}^{H} f^2\{\tfrac{1}{2}\left((X-x)^2+(Y-y)^2\right)-zZ\}dxdydz \\ = & \int_{-L}^{L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^2\{\tfrac{1}{2}\left((X-x)^2+(Y-y)^2\right)-zZ\}\rho dXdYdZ \end{aligned} \tag{31}$$

Given $\rho = \sigma(X,Y,Z)$ satisfying (29), seek the conditions under which $\delta E = 0$ for perturbations to ρ satisfying $\rho + \delta\rho \in \mathcal{R}(\sigma)$. These can be generated by keeping ρ fixed on particles in $\mathbf{X}$ space and perturbing X and Y with a displacement field χ. The displacement must be non-divergent, so can be written as $(-\frac{\partial\psi}{\partial Y}, \frac{\partial\psi}{\partial X}, 0)$ for an arbitrary function $\psi(X,Y,Z)$, and must satisfy the periodicity condition so that $\psi(X-L,Y,Z) = \psi(X+L,Y,Z)$. The restriction that the mean Y cannot be changed is then automatically enforced. We then have

$$\delta E = \int_{-L}^{L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}((X-x)\delta X + (Y-y)\delta Y - X\delta x - Y\delta y - Z\delta z)\rho dXdYdZ \tag{32}$$

where the integration is taken over particles, so that there is no $\delta\rho$, and we have used the invariance of $\int_0^H\int_{-D}^{D}\int_{-L}^{L} x^2+y^2 dxdydz$. Eq. (32) can be considered as the sum of the change due to perturbing (X,Y,Z) with (x,y,z) fixed and vice-versa. Cullen and Purser (1989) show that solutions of (20) minimise the energy at each time instant with respect to perturbations of (x,y,z) with (X,Y,Z) fixed, so that

$$\int_{-L}^{L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-X\delta x - Y\delta y - Z\delta z)\rho dXdYdZ = 0 \tag{33}$$

Using this within (32) and substituting the definitions of Ψ and $\delta\mathbf{X}$ gives after some manipulations

$$\int_{-L}^{L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\nabla.\left(\psi\rho\frac{\partial\Psi}{\partial Y},-\psi\rho\frac{\partial\Psi}{\partial X}\right)-\psi\left(-\frac{\partial\Psi}{\partial X}\frac{\partial\rho}{\partial Y}+\frac{\partial\Psi}{\partial Y}\frac{\partial\rho}{\partial X}\right)\right)dXdYdZ=0 \tag{34}$$

Assuming that ρ vanishes at a sufficiently large $|Y|$ and requiring (34) to hold for arbitrary ψ gives

$$-\frac{\partial\Psi}{\partial X}\frac{\partial\rho}{\partial Y}+\frac{\partial\Psi}{\partial Y}\frac{\partial\rho}{\partial X}=0 \tag{35}$$

This condition is precisely that for the flow to be steady as we can see from (24). Note that the linearity of (34) in ψ means that $\delta E=0$ for a perturbation obtained as the limit of a sequence of perturbations defined by displacements ψ_n, and thus for any perturbation to ρ within $\mathcal{C}_h(\rho)$.

3.3 Stable steady states- barotropic case

First consider the barotropic case where ρ and σ are functions of X,Y only, and the energy is

$$E=\frac{1}{2}\int_{-L}^{L}\int_{-D}^{D}(x-X)^2+(y-Y)^2\rho dXdY \tag{36}$$

We seek to extremise E for $\rho\in\mathcal{C}_h(\sigma)$, i.e. $\rho\in\mathcal{C}(\sigma)$ with

$$\int_{-L}^{L}\int_{-D}^{D}Y\rho dXdY=\int_{-L}^{L}\int_{-D}^{D}Y\sigma dXdY. \tag{37}$$

We characterise steady states which are stable by requiring the stationary point of the energy to be an extremum. It is clear that the maximum energy attainable under these conditions is infinite. Generate a rearrangement by a displacement $\psi=A\sin(\pi kX/L)$, implying $\delta X=0,\delta Y=\frac{A\pi k}{L}\cos\left(\frac{\pi kX}{L}\right)$. Since $|y|<D$ for all particles, (31) shows that $E\to\infty$ as $A\to\infty$. It is therefore only meaningful to seek minimum energy states. In the stability problem for the barotropic vorticity equation, however, the maximum energy state is well defined. This is because the evolution equation is written in physical space and so the displacements have to be within the physical domain. Therefore $\psi=0$ on the domain boundaries. This problem was treated by Burton and McLeod (1991).

Given σ, we seek to minimise E for $\rho\in\mathcal{C}_h(\sigma)$. (31) shows that the minimum energy is attained by making the map from (X,Y) to (x,y) as close as possible to the identity map $x=X,y=Y$. Therefore we expect that the minimum energy will be achieved by mixing the values of σ to give a mapping ι defined by

$$\iota=1:|Y-Y_0|\le D,X\le|L|\text{ and }\iota=0\text{ elsewhere,} \tag{38}$$

with Y_0 chosen to satisfy the angular momentum constraint. Then (24) shows that the velocity will be $(-fY_0,0)$ for all (x,y). This will give the lowest energy consistent with the requirement that the mean velocity and hence angular momentum is fixed. This distribution is not always achievable by mixing the given σ, as we show in the following lemma:

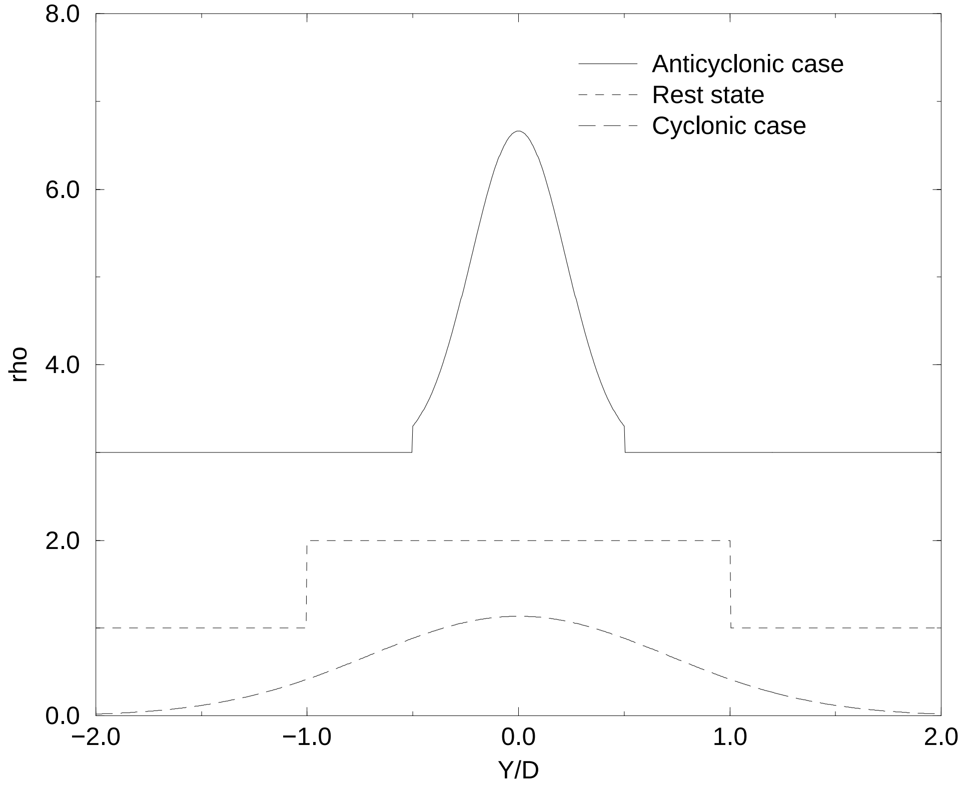

Figure 4: Graphs of $\sigma(Y)$ against Y for the cyclonic and anticyclonic cases defined in the text and the rest state $\sigma(Y) = 1, |Y| \leq D$.The base values are shifted for clarity

Lemma 4 If the support of σ has area greater than $4DL$, then the map ι given by (38) is not in $\mathcal{C}_h(\sigma)$.

Proof By assumption, there is an $\varepsilon > 0$ such that the area of the set with $\sigma > \varepsilon$ is greater than $4DL$. Then

$$\int_{-L}^{L}\int_{-\infty}^{\infty}(\sigma-\varepsilon)_+ dXdY \leq \int_{-L}^{L}\int_{-\infty}^{\infty}\sigma dXdY - 4DL\varepsilon = \int_{-L}^{L}\int_{-\infty}^{\infty}(\iota-\varepsilon)_+ dXdY.$$

This contradicts the condition (9) for ι to be in $\mathcal{C}_h(\sigma)$.

Fig. 4 shows the three possibilities for σ, assuming $Y_0 = 0$. Recalling that σ is the inverse potential vorticity, the anticyclonic case corresponds to $\sigma > 1$. The distribution shown can be mixed to give the rest state ι shown below it. The cyclonic case has $\sigma < 1$, Lemma 4 applies, and σ cannot be mixed to give ι. In that case, the minimum energy will be obtained by getting as close to ι as possible

Theorem 5 Given $\sigma(X,Y)$ satisfying (29) with compact support of area μ. If $\mu > 4DL$, the minimiser of E over $\mathcal{C}_h(\sigma)$ takes the form

$$\rho = \quad 1 : |Y - Y_0| \leq Y_1,$$

$$\begin{aligned} &= \quad \tilde{\sigma} : Y_1 < |Y - Y_0| \leq \mu/4L, \\ = 0 & \qquad\qquad \text{otherwise} \end{aligned} \tag{39}$$

where $\tilde{\sigma}$ is a monotonically decreasing function of $|Y - Y_0|$ and $0 \leq Y_1 < \mu/4L$. Y_0 is chosen to satisfy the angular momentum constraint. If $\mu \leq 4DL$ the minimiser is $\rho = \iota$, where ι is defined by (38).

Proof First prove that the mean value requirement can be separated. Choose Y_0 to satisfy

$$\int_{-L}^{L} \int_{-\infty}^{\infty} Y_0 \rho dX dY = \int_{-L}^{L} \int_{-\infty}^{\infty} (y - Y) \rho dX dY \tag{40}$$

Then set $Y = Y_0 + Y'$. Since the physical domain is centred about $y = 0$, we have $\int_{-L}^{L} \int_{-\infty}^{\infty} y \rho dX dY = 0$. (40) then shows that $\int_{-L}^{L} \int_{-\infty}^{\infty} Y' \rho dX dY = 0$. We can write the energy as

$$\begin{aligned} &\frac{1}{2} \int_{-L}^{L} \int_{-\infty}^{\infty} ((x - X)^2 + (y - Y)^2) \rho dX dY \\ = &\frac{1}{2} \int_{-L}^{L} \int_{-\infty}^{\infty} ((x - X)^2 + Y_0^2 - 2Y_0(y - Y') + (y - Y')^2) \rho dX dY \\ = &\frac{1}{2} \int_{-L}^{L} \int_{-\infty}^{\infty} ((x - X)^2 + Y_0^2 + (y - Y')^2) \rho dX dY \end{aligned} \tag{41}$$

We can therefore use Y_0 to satisfy the angular momentum constraint, and solve the energy minimisation problem for Y', with $\int_{-L}^{L} \int_{-\infty}^{\infty} Y' \rho dX dY = 0$. In the subsequent steps we assume that this has been done, and drop the primes.

The next step is to show that the minimiser is independent of X. Given σ, construct the images of the coordinate lines $y = y_n$ as curves $Y = Y_n(x)$. If Y_n is discontinuous as a function of x, the missing segment is interpolated as a straight line. Suppose that we choose $y_{-n} = -y_n$. Then

$$\int_{-L}^{L} \int_{Y_{-n}}^{Y_n} \sigma dX dY = 4L y_n \tag{42}$$

Construct a new distribution with the image of y_n being $\overline{Y}_n = \frac{1}{2L} \int_{-L}^{L} Y_n(x) dx$. Note that the averaging is in x, not X. This implies a new distribution $\overline{\sigma}$ of ρ which satisfies

$$\int_{-L}^{L} \int_{Y_{-n}}^{Y_n} \sigma dX dY = 2L \int_{Y_{-n}}^{Y_n} \overline{\sigma} dY \tag{43}$$

for all n. Thus $\overline{\sigma}$ is obtained as a local average, and Theorem 1 shows that $\overline{\sigma} \in \mathcal{C}(\sigma)$. Note that $\overline{\sigma}$ is not a strict rearrangement of σ in general, particularly if discontinuities are present. We now write the energy as an integral over physical space:

$$\frac{1}{2} \int_{-L}^{L} \int_{-D}^{D} ((x - X)^2 + (y - Y)^2) dx dy \tag{44}$$

Using the definition of $\overline{Y}_n$, we have $\int_{-L}^{L} (y_n(x) - Y_n(x)) dx = 2L(y_n - \overline{Y}_n)$. Since this holds for all n, we can make the obvious definition $\overline{Y} = \frac{1}{2L} \int_{-L}^{L} Y(x) dx$ and write $Y = \overline{Y} + Y'$. Then

$$
\begin{aligned}
&\frac{1}{2}\int_{-L}^{L}\int_{-D}^{D}\left((x-X)^2+(y-Y)^2\right)dxdy \\
=&\frac{1}{2}\int_{-L}^{L}\int_{-D}^{D}\left((x-X)^2+(y-\overline{Y})^2-2yY'+2\overline{Y}Y'+Y'^2\right)dxdy \\
=&\frac{1}{2}\int_{-L}^{L}\int_{-D}^{D}\left((x-X)^2+(y-\overline{Y})^2+Y'^2\right)dxdy
\end{aligned}
\tag{45}
$$

Eq. (45) shows that the energy is not increased by replacing Y by $\overline{Y}$, so the energy minimiser is independent of x and X.

We now have to find ρ as a function of Y only which minimises the energy. We start from a distribution $\overline{\sigma}(Y) \in \mathcal{R}_b(\sigma)$ satisfying $\int_{-\infty}^{\infty} Y\overline{\sigma}dY = 0$. This choice is not unique, but that does not affect the argument. Note first that the minimiser will be symmetric about $y=0$ and therefore about $Y=0$. This is because, given a general $Y(y)$:

$$
2\int_0^D (y-\frac{1}{2}(Y(y)-Y(-y))^2 dy \le \int_{-D}^{D}(y-Y)^2 dy \tag{46}
$$

and thus $\overline{Y} = \frac{1}{2}(Y(y)-Y(-y))$ gives a lower energy. We therefore assume that $\overline{\sigma}(Y)$ is symmetric about $Y=0$ and so work on $Y \in [0,\infty)$ only. Theorem 3 shows that y is maximised for all $Y \in [0,\infty)$ by choosing ρ to be the decreasing rearrangement of $\overline{\sigma}$ on $Y \in [0,\infty)$. There are then two cases to consider, as shown in Fig.4. First assume $\overline{\sigma}(Y) \le 1\ \forall Y$. Then $y(Y) < Y : \forall Y \in [0,\infty)$. The decreasing rearrangement of $\overline{\sigma}$ gives the largest value of y for each Y, and hence the smallest value of Y for each y. Thus it must be the minimiser of $\int(y-Y)^2dy$. This solution corresponds to (39) with $Y_1 = 0$.

Next assume that $\overline{\sigma}(Y) > 1$ for some Y. Again choose ρ to be the decreasing rearrangement of $\overline{\sigma}(Y)$, giving the maximum possible value $y_m(Y)$ of y for each Y. The assumption means that, for some Y, $y_m(Y) > Y$. Choose the largest Y, say Y_M, for which this is true. According to Theorem 1, we can replace $y(Y)$ by its linear interpolant over any range of Y while staying within $\mathcal{C}(\overline{\sigma})$. Thus if $Y_M = D$, we can choose $y(Y) = Y$, $0 \le Y \le D$, $y(Y) = D$, $Y > D$ and obtain zero energy. Otherwise, choose $y(Y) = Y$ for $Y \le Y_M$ and $y(Y) = y_m(Y)$ for $Y > Y_M$. By the definition of Y_M, $y_m(Y) < Y$ for $Y > Y_M$. Theorem 3 states that no larger values of y for each Y, and therefore no smaller values of Y for each y, can be obtained within $\mathcal{C}(\overline{\sigma})$. Therefore this choice minimises $\int(y(Y)-Y)^2dy$, as required.

Theorem 5 shows that the only minimum energy states are distributions independent of X with $\rho \le 1$ everywhere. Since ρ is an inverse potential vorticity, this condition excludes values of potential vorticity less than 1, which correspond to anticyclonic relative vorticity. This agrees with the result of Kushner and Shepherd (1995) that there were no stable shear flows with anticyclonic shear. This argument also shows that no steady states with anticyclonic relative vorticity can be stable in any limited domain with rigid boundary conditions under semi-geostrophic dynamics. They could be stable in doubly periodic flows, because there is then no region with $\rho = 0$ in the (X,Y) plane to mix with the non-zero values. Ren (2000a) discusses the physical relevance of this stability condition, suggesting that it may be most physically relevant in the baroclinic case.

3.4 Baroclinic case

If σ depends on Z, write $\mu(Z) = \frac{1}{2}\max(S(Z), 2D)$, where $2LS(Z)$ is the area over which $\sigma(X,Y,Z)$ is non-zero. Set $M = \max_Z(\mu(Z))$, for all Z. Since σ can only be rearranged on Z surfaces, we expect the energy minimiser ρ

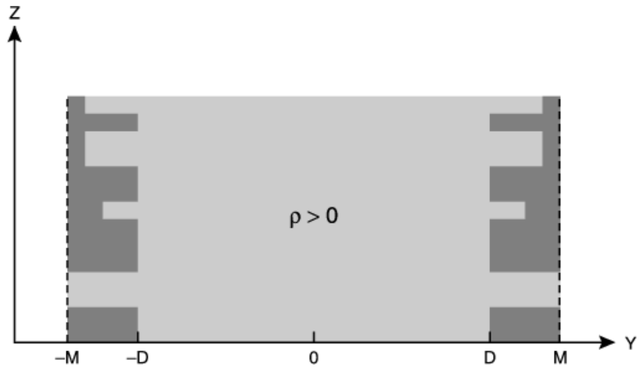

Figure 5: The region of the (Y,Z) plane for which ρ is non-zero in the energy minimising state for the two extreme cases; total shaded area: $fl/NH \gg 1$, light shaded area: $fl/NH \ll 1$.

to be obtained by first assuming zero angular momentum on each Z surface, and minimising the energy on each Z surface separately using Theorem 5. It may then be advantageous to remove the Z dependence by mixing σ uniformly over the whole region $|Y| \leq M$. The angular momentum constraint, which is vertically integrated, is then used to displace the entire solution by some distance Y_0 in Y. For each Z, the size of the set for which ρ is non-zero cannot be less than that for which σ is non-zero. However, zero values can be mixed in to increase the size of the set to $4DL$ for each Z. If $M \leq D$, σ can be mixed to give ρ as a function of Z only whose energy will be the minimum rest state potential energy. However, if $M > D$, this state is not in $\mathcal{C}_h(\sigma)$. There then has to be kinetic energy in the minimum energy state. These situations are illustrated in Fig. 5.

We formalise these procedures in:

Theorem 6 (i) The distribution $\rho =$ constant, $|Y| \leq D$, $= 0$, $|Y| > D$ is in $\mathcal{C}_h(\sigma)$, as defined in (30), if $M \leq D$. If so, then this distribution minimises the energy (31) over $\mathcal{C}_h(\sigma)$ and the geostrophic wind takes a uniform value on Ω. This value is equal to the minimum rest state potential energy if the specified angular momentum is also zero.

(ii) If the specified angular momentum is zero, the minimum possible kinetic energy achievable for $\rho \in \mathcal{C}_h(\sigma)$ is $\int_{-\infty}^{\infty} 4L \int_D^{\mu(Z)} (D - Y^*)^2 \rho^* dY dZ$, where $\rho^*(Y)$ is the decreasing rearrangement of $\overline{\sigma}(Y)$ for each Z and $dY^*/dy = (\rho^*)^{-1}$. $\overline{\sigma}(Y)$ is as defined in the previous subsection.

(iii) The potential energy is minimised for $\rho \in \mathcal{C}_h(\sigma)$ by choosing ρ to take a uniform value for $|Y - Y_0| \leq M$ for each Z.

Proof (i) Theorem 1 applied at each value of Z shows that replacing σ by its mean value over its support gives a member ρ of $\mathcal{C}_h(\sigma)$. If $S(Z) < 2D$, we can then average ρ over a set with area $4DL$ which includes its support.

This area can then be rearranged to be the set $(X,Y) \in (-L,L) \times (-D,D)$. The angular momentum is then zero, and the constraint can be satisfied by translating ρ to be non-zero on the set $(X,Y) \in (-L,L) \times (-D+Y_0, D+Y_0)$ for each Z. ρ is now a function of Z only. The potential energy $\int_\Omega -zZdxdydz$ is minimised, as in Theorem 2, by choosing Z to be a monotonically increasing function of z. The geostrophic wind is equal to $-fY_0$ for all $(x,y,z) \in \Omega$.

(ii) We start by assuming a distribution $\overline{\sigma}(Y)$ which is independent of X and symmetric about $Y=0$. If no more information is given about $\overline{\sigma}$, we can only state that kinetic energy is implied by that part of the support of $\rho \in \mathcal{C}_h(\sigma)$ which has $|Y| > D$. Since $\overline{\sigma}$ cannot be rearranged except on Z surfaces, this is the region $D < |Y| \leq \mu(Z)$ for each Z. Since $|y| \leq D$ for all Y, the kinetic energy associated with this region is at least $4L\int_D^{\mu(Z)}(Y-D)^2\rho(Y)dY$. According to Theorem 2, this quantity will be minimised by choosing ρ to be the monotonically decreasing rearrangement of $\overline{\sigma}$. The arguments used to prove Theorem 5 show that no lower estimate can be produced if we start from a σ which depends on X and is not symmetric about $Y=0$.

(iii) By Theorem 2, and because the integral of ρ cannot be changed on any Z surface, the potential energy is minimised by taking z to be a monotonic function of Z only. The choice of ρ in (iii) above achieves this because ρ depends on Z only within $|Y| \leq M$, and ρ is zero elsewhere. Thus this choice has potential energy equal to the minimum possible value.

If the rest state distribution $\{\rho = \text{ constant}, \ |Y| \leq D, \ = 0, \ |Y| > D\} \notin \mathcal{C}_h(\sigma)$, then the solution will contain kinetic energy even if the specified angular momentum is zero. The solution will be scale-dependent, according to whether potential or kinetic energy perturbations contribute more to the total energy. Since the mapping to physical space generates a geostrophic and hydrostatic state with geopotential ϕ, the kinetic energy is of order $(fl)^{-2}$ and the excess potential energy is of order N^2h^2, where l and h are the horizontal and vertical length scales of the perturbations to the rest state geopotential, and N^2 is the Brunt-Vaisala frequency $\frac{g}{\theta_0}\frac{\partial\theta}{\partial z}$. Thus if $fl/Nh \ll 1$, i.e. the horizontal length scale is smaller than the Rossby deformation radius, it is most important to minimise the kinetic energy. The distribution of Theorem 6(ii) requires there to be less kinetic energy than that of Theorem 6(iii), since there only has to be kinetic energy associated with values of Z for which $\mu(Z) > D$. The minimum energy state will thus be Z dependent. If $fl/Nh \gg 1$, it is most important to minimise the potential energy. This is achieved by the distribution of Theorem 6(iii). In this case the horizontal length scale of the potential vorticity perturbations is large compared with the deformation radius. The energy maximising state found by BN for the three-dimensional quasi-geostrophic case also minimises the z variation of the potential vorticity distribution, and is thus like our result for large horizontal scales.

In general, we have shown that there are minimum energy states with more energy than the rest state potential energy. This gives more control over the possible dynamic evolution of the system, since only the excess energy above this minimum value is available for transient motion.

4 Discussion

We have used rearrangement theory to obtain nonlinearly stable states by extremising energy subject to potential vorticity displacements. The method allows non-smooth variations of potential vorticity, and allows weak limits corresponding to solutions with filamentary potential vorticity structure to be treated rigorously. We therefore do not have to extremise subject to a constraint like enstrophy conservation, which would not be robust in real flow due to small scale mixing. We have applied the methods within semi-geostrophic theory, though they could, as in BN, be applied to other balanced models. In semi-geostrophic theory this approach avoids the difficulties caused to other methods by the nonlinear form of potential vorticity. In many cases there are no non-trivial stable steady states which are energy minimisers. The interesting results are when there are. In the shear flow case, stable steady states may have kinetic energy and, on scales not too large compared with the deformation radius, also be baroclinic.

Possible applications include the problem of minimising the energy of the flow on the sphere. The recent generalisation of semi-geostrophic theory to the sphere by Cullen and Douglas (1999), which retains potential vorticity conservation, could be used for this purpose. Another is the problem of minimising the energy of the flow in a circular ocean basin over circularly symmetric bottom topography. This is a version of a classical problem discussed, amongst others, by Holloway (1986). An alternative problem in the same context is to minimise the potential enstrophy while keeping the energy fixed. This was shown by Adcock and Marshall (1999) to be a good way of predicting the natural attracting state of general unsteady flow over bottom topography. It would also be interesting, following BN, to analyse locally stable states using semi-geostrophic theory. As they discuss, these states may well be good models of 'blocking' patterns.

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