Nonlinear Interactions of Inhomogeneous Random Water Waves

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Abstract

A new system of equations for the two point spectral correlation functions is derived. The new equations are used to demonstrate the instability of homogeneous wave fields to inhomogeneous disturbances.

1. Introduction

Starting with investigations of Phillips (1960) and Hasselmann (1962, 1963) there has been much interest in the energy transfer due to 4-wave interactions in a nearly homogeneous random sea. Longuet-Higgins (1976) derived the narrowband limit of Hasselmann's equation by starting from the nonlinear Schrödinger equation, describing the evolution of the envelope of a narrowband, weakly nonlinear wavetrain. All this nonlinear energy transfer occurs on a rather long timescale since the rate of change of the action density C is proportional to C^3. Hence

$$(\partial C/\partial t)/C = 0(\varepsilon^4\omega_0) \tag{1.1}$$

where ε is the wave steepness and ω_0 is a typical frequency of the wave field.

A much faster energy transfer is possible in the presence of spatial inhomogeneities. For an inhomogeneous random sea Watson & West (1975) and Willebrand (1975) obtained some lower-order corrections to the transport equation of Hasselmann. Also, Alber (1978) and Alber and Saffman (1978) derived an equation describing the evolution of a random narrowband wavetrain. Just like Longuet-Higgins (1976), their starting point was the nonlinear Schrödinger equation. Finally, starting from the Zakharov's (1968) equation, Crawford, Saffman, and Yuen (1980) obtained a unified equation for the evolution of a random field of deep-water waves which accounts for both the effects of spatial inhomogeneity and the energy transfer associated with a homogeneous spectrum. From their analysis it became apparent that the spatial inhomogeneities gave rise to a much faster energy transfer:

$$(\partial C/\partial t)/C = 0(\varepsilon^2\omega_0), \tag{1.2}$$

although this energy transfer is reversible. The energy transfer associated with a homogeneous sea is, however, irreversible. It should be emphasized that the assumption of an inhomogeneous wave field makes sense because Alber (1978) showed that a homogeneous spectrum is unstable to long-wavelength perturbations if the width of the spectrum is sufficiently small. For a Gaussian spectrum the instability criterion was $\sigma_\omega/\omega_0 < \varepsilon$, where σ_ω is the width in frequency space. Similar results were also found by Crawford el al (1980) for a Lorentzian shape of the spectrum. In the limit of vanishing bandwidth the deterministic result of Benjamin and Fair (1967) on the instability of a uniform wavetrain was recovered.

Still the question remains as to whether a random field of surface gravity waves has to be regarded as inhomogeneous or not.*

The aim of this paper is to demonstrate, by way of a counter example, that the instability of homogeneous random wave fields to inhomogeneous disturbances is not limited to narrow spectra. The implication of such a conclusion is far-reaching, since it renders the current way nonlinear interactions are treated in wave forecasting models inadequate. More adequate models, from the deterministic and the stochastic points of view are given in sections 2 and 3, respectively.

The formulation in Fourier space, of the linear stability of homogenous spectra to inhomogeneous disturbances is given in section 4. Two examples, one for a narrow-spectrum, and the other for a bimodal spectrum are calculated in sections 5 and 6.

2. Deterministic equations

Our starting point is the discretized Zakharov equation, recently obtained by Rasmussen and Stiassnie (1999):

$$
\begin{aligned}
& i\frac{\partial B_{M,N}}{\partial t} + i\boldsymbol{c}_g \cdot \nabla B_{M,N} - \\
& -\frac{g}{8k_{M,N}\omega_{M,N}}\left(\frac{M^2-2N^2}{M^2+N^2}\frac{\partial^2 B_{M,N}}{\partial x^2} + \frac{6MN}{M^2+N^2}\frac{\partial^2 B_{M,N}}{\partial x \partial y} + \frac{N^2-2M^2}{M^2+N^2}\frac{\partial^2 B_{M,N}}{\partial y^2}\right) = \\
& = \Sigma_{m_1 n_1}\Sigma_{m_2 n_2}\Sigma_{m_3 n_3} T\left(\boldsymbol{k}_{M,N}, \boldsymbol{k}_{m_1,n_1}, \boldsymbol{k}_{m_2,n_2}, \boldsymbol{k}_{m_3,n_3}\right) B^*_{m_1,n_1} B_{m_2,n_2} B_{m_3,n_3} \cdot \\
& \cdot \delta_K\left(\boldsymbol{k}_{M,N} + \boldsymbol{k}_{m_1,n_1} - \boldsymbol{k}_{m_2,n_2} - \boldsymbol{k}_{m_3,n_3}\right) e^{i(\omega_{M,N}+\omega_{m1,n1}-\omega_{m2,n2}-\omega_{m3,n3})t} \quad ; \; M,N = \pm 1, \pm 2, \ldots
\end{aligned}
\tag{2.1}
$$

where

$$
\boldsymbol{k}_{m,n} = \begin{pmatrix} m\Delta \\ n\Delta \end{pmatrix} \tag{2.2}
$$

are discrete wave-numbers. m and n are integers, and Δ is the increment of a rectangular mesh in the wave-number plane. δ_K denotes Kroneckr's delta. $\boldsymbol{c}_g$ is the group velocity. The angular frequency ω is given by $\omega^2 = gk$, g being the acceleration due to gravity. The free surface elevation $\eta(\boldsymbol{x},t)$ and the velocity potential at the free surface $\psi(\boldsymbol{x},t)$ are related to the spectral amplitude functions $B_{m,n}(\boldsymbol{x},t)$ through

$$
\eta(\boldsymbol{x},t) = \frac{1}{2\pi}\sum_{m,n}\sqrt{\frac{\omega_{m,n}}{2g}}\left(B_{m,n}e^{i(k_{m,n}\cdot x - \omega_{m,n}t)} + B^*_{m,n}e^{-i(k_{m,n}\cdot x - \omega_{m,n}t)}\right) \tag{2.3}
$$

* Most of the background material up to this point was taken from Janssen (1983).

$$\psi(x,t)=\frac{-i}{2\pi}\sum_{m,n}\sqrt{\frac{g}{2\omega_{m,n}}}\left(B_{m,n}e^{i(k_{m,n}\cdot x-\omega_{m,n}t)}-B^*_{m,n}e^{-i(k_{m,n}\cdot x-\omega_{m,n}t)}\right) \tag{2.4}$$

and the opposite relation

$$B_{m,n}=\frac{\Delta^2}{2\pi}\int_A\left(\sqrt{\frac{g}{2\omega_{m,n}}}\eta(\tilde{x},t)+i\sqrt{\frac{\omega_{m,n}}{2g}}\psi(\tilde{x},t)\right)e^{-i(k_{m,n}\cdot\tilde{x}-\omega_{m,n}t)}d\tilde{x} \tag{2.5}$$

A is a square with center x and side length $2L=2\pi/\Delta$, so that the inside of this square is defined by $\tilde{x}\in(x-L,x+L)$ and $\tilde{y}\in(y-L,y+L)$.

Rasmussen and Stiassnie define two small parameters. One is the wave-number resolution parameter $\delta=\Delta/k_p$ and the other is a nonlinearity measure $\varepsilon=a$; where k_p and a are typical wave-number and typical amplitude, respectively. Here we treat the case $\delta/\varepsilon=O(1)$, for which the dispersive terms in (2.1) are of the same order as the nonlinear term. Alternative formulations are discussed in Appendix A.

3. Stochasic equations

It is assumed that the system (2.1) for the spectral amplitude functions $B_{M,N}(x,t)$ describes the evolution of the wave-field also when $B_{M,N}$ are random functions. For waves undergoing weak nonlinear interactions we follow Alber (1978) and seek a system of equations for the slow variation of the two-point space correlation spectral functions.

$$\rho_{M,N}(x_1,x_2,t)=\langle B_{M,N}(x_1,t)B^*_{M,N}(x_2,t)\rangle \tag{3.1}$$

where superscript * denotes the complex conjugate. In eq. (3.1), the angle brackets denote an ensemble average. We write eq. (2.1) at the point $x_1=(x_1,y_1)$, multiply it by $B^*_{M,N}(x_2)$ and add it to the equation for $B^*_{M,N}(x_2)$ multiplied by $B_{M,N}(x_1)$, and take ensemble average. The resulting equation is

$$\begin{aligned}
&i\frac{\partial}{\partial t}\langle B_{M,N}(x_1)B^*_{M,N}(x_2)\rangle+ic_{gM,N}\cdot(\nabla_{x_1}+\nabla_{x_2})\langle B_{M,N}(x_1)B^*_{M,N}(x_2)\rangle-\\
&-\frac{g}{8k_{M,N}\omega_{M,N}}\left[\frac{M^2-2N^2}{M^2+N^2}\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right)+\frac{6MN}{M^2+N^2}\left(\frac{\partial^2}{\partial x_1\partial y_1}-\frac{\partial^2}{\partial x_2 y_2}\right)+\right.\\
&\left.+\frac{N^2-2M^2}{M^2+N^2}\left(\frac{\partial^2}{\partial y_1^2}-\frac{\partial^2}{\partial y_2^2}\right)\right]\langle B_{M,N}(x_1)B^*_{M,N}(x_2)\rangle=\\
&=\sum_{m_1,n_1}\sum_{m_2,n_2}\sum_{m_3,n_3}T(k_{M,N},k_{m_1,n_1},k_{m_2,n_2},k_{m_3,n_3})\delta_K(k_{M,N}+k_{m_1,n_1}-k_{m_2,n_2}-k_{m_3,n_3})\\
&\cdot\left\{e^{i(\omega_{M,N}+\omega_{m_1,n_1}-\omega_{m_2,n_2}-\omega_{m_3,n_3})t}\langle B^*_{M,N}(x_2)B^*_{m_1,n_1}(x_1)B_{m_2,n_2}(x_1)B_{m_3,m_3}(x_1)\rangle-\right.\\
&\left.-e^{-i(\omega_{M,N}+\omega_{m_1,n_1}-\omega_{m_2,n_2}-\omega_{m_3,n_3})t}\langle B_{M,N}(x_1)B_{m_1,n_1}(x_2)B^*_{m_2,n_2}(x_2)B^*_{m_3,m_3}(x_2)\rangle\right\}
\end{aligned} \tag{3.2}$$

The derivatives with respect to x_1, y_1, x_2 and y_2 can be replaced by derivatives with respect to the average coordinates

$$x=\frac{1}{2}(x_1+x_2), \quad y=\frac{1}{2}(y_1+y_2) \qquad ; \tag{3.3}$$

and with respect to the spatial separation coordinates

$$r_x = x_1 - x_2, \quad r_y = y_1 - y_2 \tag{3.4}$$

Thus from (3.2) we obtain for $\rho_{M,N}(\boldsymbol{x},\boldsymbol{r},t)$:

$$\begin{aligned}
& i\frac{\partial \rho_{M,N}}{\partial t} + i c_{gM,N}\cdot\nabla_x \rho_{M,N} - \frac{g}{4k_{M,N}\omega_{M,N}}\left[\frac{M^2-2N^2}{M^2+N^2}\frac{\partial^2\rho_{M,N}}{\partial x\partial r_x} + \right.\\
& \left. + \frac{3MN}{M^2+N^2}\left(\frac{\partial^2\rho_{M,N}}{\partial x\partial r_y}+\frac{\partial^2\rho_{M,N}}{\partial y\partial r_x}\right)+\frac{N^2-2M^2}{M^2+N^2}\frac{\partial^2\rho_{M,N}}{\partial y\partial r_y}\right] = \\
& = \sum_{m_1,n_1}\sum_{m_2,n_2}\sum_{m_3,n_3} T\left(\boldsymbol{k}_{M,N},\boldsymbol{k}_{m_1,n_1},\boldsymbol{k}_{m_2,n_2},\boldsymbol{k}_{m_3,n_3}\right)\delta_K\left(\boldsymbol{k}_{M,N}+\boldsymbol{k}_{m_1,n_1}-\boldsymbol{k}_{m_2,n_2}-\boldsymbol{k}_{m_3,n_3}\right)\cdot \\
& \cdot\left\{e^{i\left(\omega_{M,N}+\omega_{m_1,n_1}-\omega_{m_2,n_2}-\omega_{m_3,n_3}\right)t}\left\langle B^*_{M,N}(\boldsymbol{x}_2)B^*_{m_1,n_1}(\boldsymbol{x}_1)B_{m_2,n_2}(\boldsymbol{x}_1)B_{m_3,m_3}(\boldsymbol{x}_1)\right\rangle - \right.\\
& \left. - e^{-i\left(\omega_{M,N}+\omega_{m_1,n_1}-\omega_{m_2,n_2}-\omega_{m_3,n_3}\right)t}\left\langle B_{M,N}(\boldsymbol{x}_1)B_{m_1,n_1}(\boldsymbol{x}_2)B^*_{m_2,n_2}(\boldsymbol{x}_2)B^*_{m_3,m_3}(\boldsymbol{x}_2)\right\rangle\right\}
\end{aligned} \tag{3.5}$$

As seen in (3.5) the evolutionary equation for the second-order correlation involves fourth-order correlation terms. To evaluate these terms, we assume that $B_{M,N}(\boldsymbol{x},t)$ correspond initially to a Gaussian random process, and we further assume that the evolving random amplitudes retain the same Gaussian statistical properties. For Gaussian statistics, the fourth-order cumulant vanishes, allowing us to write the fourth-order correlation in terms of products of pairs of second-order correlations, i.e.

$$\begin{aligned}
& \left\langle B^*_{M,N}(\boldsymbol{x}_2)B^*_{m_1,n_1}(\boldsymbol{x}_1)B_{m_2,n_2}(\boldsymbol{x}_1)B_{m_3,m_3}(\boldsymbol{x}_1)\right\rangle = \\
& = 2\left\langle B^*_{M,N}(\boldsymbol{x}_2)B_{m_2,n_2}(\boldsymbol{x}_1)\right\rangle\left\langle B^*_{m_1,n_1}(\boldsymbol{x}_1)B_{m_3,n_3}(\boldsymbol{x}_1)\right\rangle\delta_K\left(\boldsymbol{k}_{M,N}-\boldsymbol{k}_{m_2,n_2}\right)\delta_K\left(\boldsymbol{k}_{m_1,n_1}-\boldsymbol{k}_{m_3,m_3}\right)
\end{aligned} \tag{3.6}$$

A similar expression can be written for the other fourth-order correlation in (3.5). Under the Gaussian closure approximation then, eq. (3.5) can be written

$$\begin{aligned}
& i\frac{\partial \rho_{M,N}}{\partial t} + i c_{gM,N}\cdot\nabla_x \rho_{M,N} - \frac{g}{4k_{M,N}\omega_{M,N}}\left[\frac{M^2-2N^2}{M^2+N^2}\frac{\partial^2\rho_{M,N}}{\partial x\partial r_x} + \right.\\
& \left. + \frac{3MN}{M^2+N^2}\left(\frac{\partial^2\rho_{M,N}}{\partial x\partial r_y}+\frac{\partial^2\rho_{M,N}}{\partial y\partial r_x}\right)+\frac{N^2-2M^2}{M^2+N^2}\frac{\partial^2\rho_{M,N}}{\partial y\partial r_y}\right] = \\
& = 2\rho_{M,N}\sum_{m,n} T\left(\boldsymbol{k}_{M,N},\boldsymbol{k}_{m,n},\boldsymbol{k}_{M,N},\boldsymbol{k}_{m,n}\right)\left[C_{m,n}\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{r}\right)-C_{m,n}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{r}\right)\right]
\end{aligned} \tag{3.7}$$

where

$$C_{m,n}(\boldsymbol{x},t)=\rho_{m,n}(\boldsymbol{x},0,t) \tag{3.8}$$

is the wave action spectral function.

4. Formulation of a linear-stability problem

In order to study the stability of homogenous spectra to inhomogeneous disturbances it is convenient to transform (3.7) from the physical to the Fourier plane.

First, we define

$$\hat{\rho}_{M,N}(\boldsymbol{q},\boldsymbol{r},t)=\frac{1}{2\pi}\int \rho_{M,N}(\boldsymbol{x},\boldsymbol{r},t)e^{-i\boldsymbol{q}.\boldsymbol{x}}d\boldsymbol{x} \tag{4.1}$$

and take the $\boldsymbol{x}$ to $\boldsymbol{q}$ Fourier transform of (3.7):

$$\begin{aligned}
&i\frac{\partial\hat{\rho}_{M,N}}{\partial t}-\left(\boldsymbol{c}_{gM,N}\cdot\boldsymbol{q}\right)\hat{\rho}_{M,N}-\frac{gi}{4k_{M,N}\omega_{M,N}}\left[\frac{M^2-2N^2}{M^2+N^2}q_x\frac{\partial\hat{\rho}_{M,N}}{\partial r_x}+\right.\\
&\left.+\frac{3MN}{M^2+N^2}\left(q_x\frac{\partial\hat{\rho}_{M,N}}{\partial r_y}+q_y\frac{\partial\hat{\rho}_{M,N}}{\partial r_x}\right)+\frac{N^2-2M^2}{M^2+N^2}q_y\frac{\partial\hat{\rho}_{M,N}}{\partial r_y}\right]=\\
&=\frac{1}{\pi}\sum_{m,n}T_{(M,N),(m,n)}\int\hat{\rho}_{M,N}(\boldsymbol{q}_1)\hat{C}_{m,n}(\boldsymbol{q}-\boldsymbol{q}_1)\left[e^{i(\boldsymbol{q}-\boldsymbol{q}_1)\cdot\frac{\boldsymbol{r}}{2}}-e^{-i(\boldsymbol{q}-\boldsymbol{q}_1)\cdot\frac{\boldsymbol{r}}{2}}\right]d\boldsymbol{q}_1
\end{aligned} \tag{4.2}$$

where $T_{(M,N)(m,n)}=T(\boldsymbol{k}_{M,N,}\boldsymbol{k}_{m,n,}\boldsymbol{k}_{M,N,}\boldsymbol{k}_{m,n})$.

Next we define

$$\hat{\hat{\rho}}_{M,N}(\boldsymbol{q},\boldsymbol{p},t)=\frac{1}{2\pi}\int\hat{\rho}_{M,N}(\boldsymbol{q},\boldsymbol{r},t)e^{-i\boldsymbol{p}.\boldsymbol{r}}d\boldsymbol{r}, \tag{4.3}$$

and take the $\boldsymbol{r}$ to $\boldsymbol{p}$ Fourier transform of (4.2):

$$\begin{aligned}
&i\frac{\partial\hat{\hat{\rho}}_{M,N}}{\partial t}-\left(\boldsymbol{c}_{gM,N}.\boldsymbol{q}\right)\hat{\hat{\rho}}_{M,N}+\frac{g}{4k_{M,N}\omega_{M,N}}\left[\frac{M^2-2N^2}{M^2+N^2}q_xp_x+\frac{3MN}{M^2+N^2}\left(q_xp_y+q_yp_x\right)+\right.\\
&\left.+\frac{N^2-2M^2}{M^2+N^2}q_yp_y\right]\hat{\hat{\rho}}_{M,N}=\frac{1}{\pi}\sum_{m,n}T_{(M,N),(m,n)}\int\hat{C}_{m,n}(\boldsymbol{q}-\boldsymbol{q}_1)\cdot\\
&\cdot\left[\hat{\hat{\rho}}_{M,N}\left(\boldsymbol{q}_1,\boldsymbol{p}-\frac{1}{2}(\boldsymbol{q}-\boldsymbol{q}_1),t\right)-\hat{\hat{\rho}}_{M,N}\left(\boldsymbol{q}_1,\boldsymbol{p}+\frac{1}{2}(\boldsymbol{q}-\boldsymbol{q}_1),t\right)\right]d\boldsymbol{q}_1
\end{aligned} \tag{4.4}$$

However

$$\hat{C}_{m,n}(q)=\frac{1}{2\pi}\int\hat{\hat{\rho}}_{m,n}(q,p_1)dp_1 \quad , \tag{4.5}$$

so that finally (3.7)reduces to

$$i\frac{\partial\hat{\hat{\rho}}_{M,N}}{\partial t}-\left(c_{gM,N}\cdot q\right)\hat{\hat{\rho}}_{M,N}+\frac{g}{4k_{M,N}\omega_{M,N}}\left[\frac{M^2-2N^2}{M^2+N^2}q_x p_x+\frac{3MN}{M^2+N^2}\left(q_x p_y+q_y p_x\right)+\right.$$
$$\left.+\frac{N^2-2M^2}{M^2+N^2}q_y p_y\right]\hat{\hat{\rho}}_{M,N}=\frac{1}{2\pi^2}\sum_{m,n}T_{(M,N),(m,n)}\iint\hat{\hat{\rho}}_{m,n}(q-q_1,p_1)\cdot \tag{4.6}$$
$$\cdot\left[\hat{\hat{\rho}}_{M,N}\left(q_1,p-\frac{1}{2}(q-q_1),t\right)-\hat{\hat{\rho}}_{M,N}\left(q_1,p+\frac{1}{2}(q-q_1),t\right)\right]dq_1dp_1$$

For a homogeneous ocean $\rho_{m,n}$ is independent of x and we denote it as $\rho^{(h)}_{m,n}(r,t)$, so that its x to q and r to p Fourier transform is $2\pi\hat{\rho}^{(h)}(p)\delta(q)$. From eq. (3.7) or (4.6) one can see that any homogeneous wave-field is necessarily also stationary. The main issue of this note is to address the question of stability of the homogeneous solution to a small inhomogeneous disturbance $\rho^{(d)}_{m,n}(x,r,t)$.

Substituting

$$\hat{\hat{\rho}}_{m,n}(q,p,t)=2\pi\hat{\hat{\rho}}^{(h)}_{m,n}(p)\delta(q)+\hat{\hat{\rho}}^{(d)}_{m,n}(q,p,t), \tag{4.7}$$

into(4.6), and linearizing in $\hat{\hat{\rho}}^{(d)}_{m,n}$, we obtain:

$$i\frac{\partial}{\partial t}\hat{\hat{\rho}}^{(d)}_{M,N}+\{-c_{gM,N}.q+\frac{g}{4k_{M,N}\omega_{M,N}}[\frac{M^2-2N^2}{M^2+N^2}q_x p_x+\frac{3MN}{M^2+N^2}(q_x p_y+q_y p_x)+$$
$$+\frac{N^2-2M^2}{M^2+N^2}q_y p_y]\}\hat{\hat{\rho}}^{(d)}_{M,N}=\frac{1}{\pi}[\hat{\hat{\rho}}^{(h)}_{M,N}(p-\frac{q}{2})-\hat{\hat{\rho}}^{(h)}_{M,N}(p+\frac{q}{2}]\sum_{m,n}\mathrm{T}_{(M,N),(m,n)}\int\hat{\hat{\rho}}^{(d)}_{m,n}(q,p_1)dp_1 \tag{4.8}$$

Assuming disturbances with wave number $q^{(d)}$ and frequency Ω:

$$\hat{\hat{\rho}}^{(d)}{}_{M,N}=f_{M,N}(p)e^{-i(\Omega+C_{gM,N}.q^{(d)})t}\delta(q-q^{(d)}) \tag{4.9}$$

and substitution in (4.8) gives:

$$\{\Omega+\frac{g}{4k_{M,N}\omega_{M,N}}[\frac{M^2-2N^2}{M^2+N^2}q^{(d)}_x p_x+\frac{3MN}{M^2+N^2}(q^{(d)}_x p_y+q^{(d)}_y p_x)+\frac{N^2-2M^2}{M^2+N^2}q^{(d)}_y p_y]\}f_{m,n}(p)=$$
$$=\frac{1}{\pi}[\hat{\hat{\rho}}^{(h)}_{M,N}(p-\frac{q^{(d)}}{2})-\hat{\hat{\rho}}^{(h)}_{M,N}(p+\frac{q^{(d)}}{2})]\sum_{m,n}\mathrm{T}_{(M,N),(m,n)}\int f_{m,n}(p_1)dp_1 \tag{4.10}$$

Dividing eq. (4.10) by the term in the curly brackets, integrating over p, and defining a new variable

$$\alpha_{m,n} = \int f_{m,n}(p_1)dp_1, \tag{4.11}$$

the following algebric linear system is obtained

$$\alpha_{M,N} = I_{M,N} \sum_{m,n} T_{(M,N),(m,n)} \alpha_{m,n} \tag{4.12}$$

where

$$\mathrm{I}_{M,N} = \frac{1}{\pi} \int \frac{[\hat{\rho}^{(h)}_{M,N,}(p - \frac{q^{(d)}}{2}) - \hat{\rho}^{(h)}_{M,N}(p + \frac{q^{(d)}}{2})]dp}{\Omega + \frac{g}{4k_{M,N}\omega_{M,N}}[\frac{M^2 - 2N^2}{M^2 + N^2} q_x^{(d)} p_x + \frac{3MN}{M^2 + N^2}(q_x^{(d)} p_y + q_y^{(d)} p_x) + \frac{N^2 - 2M^2}{M^2 + N^2} q_y^{(d)} p_y]} \tag{4.13}$$

In Appendix B we show that $\hat{\rho}^{(h)}_{M,N}$ has the following relation to the wavenumber spectrum $S_{\eta\eta}$:

$$\hat{\rho}_{M,N}(p) = \frac{4\pi^3 g}{L^4 \omega_{M,N}} \frac{sin^2(p_x L)}{p_x^2} \frac{sin^2(p_y L)}{p_y^2} \left[1 + \frac{\omega^2_{M,N}}{\omega^2(k_{M,N} + p}\right] S_{\eta\eta}(p + k_{M,N}) \tag{4.14}$$

For simplicity we assume the homogeneous spectrum to be discrete, so that

$$S_{\eta\eta}(k) = \sum_{m,n} S_{m,n}\delta(k - k_{m,n}) \tag{4.15}$$

In the sequel we'll study two particular cases. A unimodel spectrum, for which

$$S_{\eta\eta} = S_o \delta(k - k_o) \quad , \qquad k_o = (k_o, 0) \tag{4.16}$$

and the bimodal case

$$S_{\eta\eta} = S_I \delta(k - k_I) + S_{II} \delta(k - k_{II}), \tag{4.17}$$

5. The instability of a unimodal spectrum

Substituting (4.16) into (4.14) gives

$$\hat{\rho}^{(h)}_o(p) = \frac{8\pi^3 g S_0}{\omega_o} \delta(p), \tag{5.1}$$

so that (4.13) becomes

$$\mathrm{I}_o = \frac{-2\pi^2 g^2 S_o}{k_o \omega_o^2}[q_x^{(d)2} - 2q_y^{(d)2}]/[\Omega^2 - (\frac{g}{8k_o\omega_o})^2 (q_x^{(d)2} - 2q_y^{(d)2})^2] \tag{5.2}$$

Equation (4.12) degenerates into

$$\mathrm{I}_o \mathrm{T}_o = 1 \tag{5.3}$$

with

$$\mathrm{T}_o = k_o^{\ 3} / 4\pi^2 \tag{5.4}$$

Solving for Ω we obtain

$$\Omega^2 = \frac{g}{8^2 k_o^{\ 3}}(q_x^{(d)2} - 2q_y^{(d)2})(q_x^{(d)2} - 2q_y^{(d)2} - 32k_o^{\ 4} S_o) \tag{5.5}$$

For stability we need $\Omega^2 > 0$. If $q_x^{(d)2} < 2q_y^{(d)2}$ both brackets in (5.5)are negative and the situation is stable. However, for

$$q_x^{(d)2} > 2q_y^{(d)2}, \text{ and}$$

$$k_o^4 S_o > \frac{1}{32}(q_x^{(d)2} - 2q_x^{(d)2}) \tag{5.6}$$

a homogenous sea is unstable to the inhogenous disturbances.

For unidirectional seas, i.e $q_y^{(d)} = 0$, equation (5.6) gives

$$\left(q_x^{(d)} / k_o\right)^2 < 32\left(k_o^2 \quad S_o\right) \tag{5.7}$$

The most unstable disturbance and its growth rate are

$$(q_{max} / k_o)^2 = 16\left(k_o^2 \ S_o\right) \quad , \qquad 5.8)$$

and

$$\Omega_{max} / \sqrt{gk_o} = 2\left(k_o^2 S_o\right) \tag{5.9}$$

which is the stochastic manifestation of the Benjamin and Feir instability.

6. The instability of a bimodal spectrum

For the bimodal sea given by (4.17), equation (4.12) gives

$$\alpha_{\mathrm{I}} = \mathrm{I}_{\mathrm{I}}(\mathrm{T}_{\mathrm{I}}\alpha_{\mathrm{I}} + \mathrm{T}_{\mathrm{I,II}}\alpha_{\mathrm{II}}) \tag{6.1$_{\mathrm{I}}$}$$

$$\alpha_{\mathrm{II}} = \mathrm{I}_{\mathrm{II}}(\mathrm{T}_{\mathrm{II,I}}\alpha_{\mathrm{I}} + \mathrm{T}_{\mathrm{II}}\alpha_{\mathrm{II}}) \tag{6.1$_{\mathrm{II}}$}$$

To simplify the algebra we assume that the original waves, as well as the disturbances are collinear and that $k_{\mathrm{II}} > k_{\mathrm{I}}$. Under this condition

$$\mathrm{T}_{\mathrm{I}} = \frac{k_{\mathrm{I}}^{\,3}}{4\pi^2},\quad \mathrm{T}_{\mathrm{II}} = \frac{k_{\mathrm{II}}^{\,3}}{4\pi^2},\quad \mathrm{T}_{\mathrm{I,II}} = \mathrm{T}_{\mathrm{II,I}} = \frac{k_{\mathrm{I}}^2 k_{\mathrm{II}}}{4\pi^2} \tag{6.2}$$

$$\mathrm{I}_{\mathrm{I}} = \frac{-2\pi^2 g^2 S_{\mathrm{I}}}{k_{\mathrm{I}}\omega_{\mathrm{I}}^{\,2}} q^2 / [\Omega^2 - (\frac{g}{8k_{\mathrm{I}}\omega_{\mathrm{I}}})^2 q^4] \tag{6.3$_{\mathrm{I}}$}$$

$$\mathrm{I}_{\mathrm{II}} = \frac{-2\pi^2 g^2 S_{\mathrm{II}}}{k_{\mathrm{II}}\omega^2{}_{\mathrm{II}}} q^2 / [\Omega^2 - (\frac{g}{8k_{\mathrm{II}}\omega_{\mathrm{II}}})^2 q^4] \tag{6.3$_{\mathrm{II}}$}$$

For (6.1) to have a solution its determinant must vanish, i.e:

$$(\mathrm{I}_{\mathrm{I}}\mathrm{T}_{\mathrm{I}} - 1)(\mathrm{I}_{\mathrm{II}}\mathrm{T}_{\mathrm{II}} - 1) = \mathrm{T}_{\mathrm{I,II}}^2 \mathrm{I}_{\mathrm{I}}\mathrm{I}_{\mathrm{II}}; \tag{6.4}$$

which reduces into a quadratic equation for Ω^2

$$\Omega^4 + b\Omega^2 + c = 0 \tag{6.5}$$

where

$$b = (\gamma_{\mathrm{I}} S_{\mathrm{I}} + \gamma_{\mathrm{II}} S_{\mathrm{NI}}) q^2 - (\delta_{\mathrm{I}} + \delta_{\mathrm{II}}) q^4 \tag{6.6}$$

$$c = (\gamma_{\mathrm{I}}\gamma_{\mathrm{II}} - \gamma_{\mathrm{I}}^{\,2}) S_{\mathrm{I}} S_{\mathrm{II}} q^4 - (\gamma_{\mathrm{I}} S_{\mathrm{I}} \delta_{\mathrm{II}} + \gamma_{\mathrm{II}} S_{\mathrm{II}} \delta_{\mathrm{I}}) q^6 + \delta_{\mathrm{I}}\delta_{\mathrm{II}} q^8 \tag{6.7}$$

and

$$\gamma_{\mathrm{I}} = g k_{\mathrm{I}} / 2,\quad \gamma_{\mathrm{II}} = g k_{\mathrm{II}} / 2,\quad \delta_{\mathrm{I}} = \frac{g}{64 k_{\mathrm{I}}^3},\quad \delta_{\mathrm{II}} = \frac{g}{64 k_{\mathrm{II}}^3} \tag{6.8}$$

One can show that the discriminant of (6.5) is always positive, so that (6.5) has always two real roots. For stability both roots have to be positive, which requires $b < 0$ and $c > 0$.

The first condition, $b<0$, yields

$$q^2 > \frac{\gamma_{\mathrm{I}} S_{\mathrm{I}} + \gamma_{\mathrm{II}} S_{\mathrm{II}}}{\delta_{\mathrm{I}} + \delta_{\mathrm{II}}} = \frac{32 k_{\mathrm{I}}^3 k_{\mathrm{II}}^3 (k_{\mathrm{I}} S_{\mathrm{I}} + k_{\mathrm{II}} S_{\mathrm{II}})}{k_{\mathrm{I}}^3 + k_{\mathrm{II}}^3} \equiv s \tag{6.9}$$

The second condtition, $c>0$, leads to

$$0 < q^2 < s_1, \quad q^2 > s_2 > s_1 \tag{6.10}$$

where $s_{1,2}$ are solutions of $c=0$, (see 6.8), namely

$$s_{1,2} = \{\gamma_{\mathrm{I}}\delta_{\mathrm{II}} S_{\mathrm{I}} + \gamma_{\mathrm{II}}\delta_{\mathrm{I}} S_{\mathrm{II}} \pm \sqrt{(\gamma_{\mathrm{I}}\delta_{\mathrm{II}} S_{\mathrm{I}} + \gamma_{\mathrm{II}}\delta_{\mathrm{I}} S_{\mathrm{II}})^2 - 4\gamma_{\mathrm{I}}(\gamma_{\mathrm{II}} - \gamma_{\mathrm{I}})\delta_{\mathrm{I}}\delta_{\mathrm{II}} S_{\mathrm{I}} S_{\mathrm{II}}}\ \} / 2\delta_{\mathrm{I}}\delta_{\mathrm{II}} \tag{6.11}$$

both of them positive.

Thus, the region $s_1 < q^2 < s_2$ is always unstable. However the instability region may be wider, depending on the location of s compared to s_1, and s_2.

Example (i): two adjacent modes

Here we choose $k_{\mathrm{I}} = k_o$, $k_{\mathrm{II}} = k_o + \Delta$; and $S_I = \alpha S_o$, S_{II} $(1-\alpha)$ S_o. The instability range $0 < q < \sqrt{s_2}$ is given by

$$\left(q_x^{(d)} / k_\alpha\right)^2 < 32 \cdot \left[1 + \left(\alpha^2 - \alpha\right)\left(\Delta / k_\alpha\right)\right]\left(k_\alpha^2 S_o\right), \tag{6.12}$$

where the representative wave-number is

$$k_\alpha = k_o + (1-\alpha)\,\Delta \tag{6.13}$$

For $\alpha = 0$, 1 we recover the unimodal result (5.7).

For $\alpha =$ ½ the instability range, and the growth-rate are

$$\left(q_x^{(d)} / k_{1/2}\right)^2 < 32\left(1 - 0.25\Delta / k_{1/2}\right)\left(k_{1/2}^2 S_o\right), \tag{6.14}$$

and

$$\Omega_{1/2} / \sqrt{g k_{1/2}} = 2\left(1 - 0.25\Delta / k_{1/2}\right)\left(k_{1/2}^2 S_o\right) \tag{6.15}$$

Both the instability range and the growth rate coefficients are smaller than in the unimodal case (either $\alpha =$ 1, or $\alpha =$ 0), which seems to support Alber's finding that a narrow spectrum tends to become less unstable when widening.

Example (ii): two well separated modes

Here we take $k_{II} = 2k_I = 2k_o$; *and* $S_I = \alpha S_o$, $S_{II} = (1 - \alpha) S_o$. The representative wave-number is

$$k_\alpha = (2 - \alpha)k_o \tag{6.16}$$

The instability range and the most unstable mode for $\alpha = ½$ are

$$\left(q_x^{(d)} / k_{1/2}\right)^2 < \frac{128}{81}\left(17 + \sqrt{257}\right)\left(k_{1/2}^2\, S_o\right) = 52.2\left(k_{1/2}^2\, S_o\right), \tag{6.17}$$

and

$$\Omega_{1/2} / \sqrt{g k_{1/2}} = 2.17\left(k_{1/2}^2 S_o\right) \tag{6.18}$$

Note that the instability range coefficient 52.2 is significantly larger than 32, its counterpart for the unimodal case. The growth-rate coefficient 2.17 is also somewhat larger than 2, see equation (5.9). Thus, the broadening of the spectrum, in this case, seems to contradict Alber's result.

7. Discussion

Based on the examples given in section 5 and 6, we conjecture that almost any homogenous wave field is unstable to inhomogeneous disturbances. This contradicts the conclusions of previous authors that found no instability above a certain spectral width. However, their conclusion was based on a model equation, which is valid for narrow spectra only, which makes any results regarding wider spectra doubtful, if not meaningless.

If our above conjecture will find further support, which we believe it will, then the way nonlinear interaction in wave-forecasting models is treated; will require a significant change. Namely, models like our equation (3.7) for two-point spectral correlation function, will replace Hasselmann's action transfer equation.

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References:

Alber, I. E. 1978 The effects of randomness on the stability of two dimensional surface wavetrains. *Proc. R. Soc. Lond.* **A363**, 525-546.

Alber, I.E. & Saffman, P. G. 1978 Stability of random nonlinear deep water waves with finite bandwidth spectra. *TRW Defense and Space Systems Group Rep.* No. 31326-6035-RU-OO.

Benjamin, T.B. & Feir, J. E. 1967 The disintegration of wave on deep water. Part 1. Theory. *J. Fluid Mech.* **27**, 417-430.

Crawford, D.R., Saffman, P. G. & Yuen, H.C. 1980 Evolution of a random inhomogeneous field of nonlinear deep-water gravity waves. *Wave Motion* **2**, 1-16.

Hasselmann, K. 1962 On the non-linear energy transfer in a gravity-wave spectrum. Part 1. General theory. *J. Fluid Mech.* **12**, 481.

Hasselmann, K. 1963 On the non-linear energy transfer in a gravity-wave spectrum. Part 2. Conservation theorems, wave-particle analogy, irreversibility. *J. Fluid Mech.* **15**, 273.

Janssen, P. A. E. M. 1983 Long-time behavious of a random inhomogeneous field of weakly nonlinear surface gravity waves. *J. Fluid Mech.* **133**, 113-132.

Longuet-Higgins, M. S 1976 On the nonlinear transfer of energy in the peak of a gravity-wave spectrum: a simplified model. *Proc. R. Soc. Lond.* **A347**, 311-328.

Phillips, O.M. 1960 On the dynamics of unsteady gravity waves of finite amplitude. Part 1. *J. Fluid Mech.* **9**, 193-217.

Rasmussen, J. H. 1999 Deterministic and stochastic modelling of surface gravity waves in finite depth. Ph.D. Thesis, ISVA Series, Paper 68, 245 p.

Rasmussen, J. H. and Stiassnie, M. 1999 Discretization of Zakharov's equation. *Eur. J. Mech. B/Fluids*, **18**, 353-364.

Watson, K. M. & West, B.J. 1975 A transport equation description of nonlinear ocean surface wave interactions. *J. Fluid Mech.* **70**, 815-826.

Willebrand, J. 1975 Transport in a nonlinear and inhomogeneous random gravity wave field. *J. Fluid Mech.* **70**, 113-126.

Zakharov, V.E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Zh. Prikl. Mekh. Tekh. Fiz.* **9**, 86-94. English trans in *J. Appl. Mech. Tech. Phys.* **9**, 190-194.

Appendix A: Alternative formulations

Equation (3.50) in Rasmussen (1999) reads:

$$\frac{\partial B}{\partial t}+\boldsymbol{c}_g\cdot\nabla B+\frac{ig}{8k\omega}\left(\frac{k_x^2-2k_y^2}{k^2}\frac{\partial^2 B}{\partial x^2}+\frac{6k_xk_y}{k^2}\frac{\partial^2 B}{\partial x\partial y}+\frac{k_y^2-2k_x^2}{k^2}\frac{\partial^2 B}{\partial y^2}\right)=$$

$$=-i\iiint_{-\infty}^{\infty} T_{0,1,2,3}\,B_1^*B_2B_3\,e^{i(\omega+\omega_1-\omega_2-\omega_3)t}\delta_{0+1-2-3}d\boldsymbol{k}_1d\boldsymbol{k}_2d\boldsymbol{k}_3 \tag{A.1}$$

where

$$B(\boldsymbol{k},\boldsymbol{x}_1,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sqrt{\frac{g}{2\omega}}\,\eta(\boldsymbol{x}_o,\boldsymbol{x}_1)+i\sqrt{\frac{\omega}{2g}}\,\psi(\boldsymbol{x}_o,\boldsymbol{x}_1)\right)e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}_0-\omega t)}d\boldsymbol{x}_o \tag{A.2}$$

In the derivation procedure of (A.1), Rasmussen follows similar steps as for the Zakharov equation, but takes into account multiple horizontal scales ($\boldsymbol{x}_o$, $\boldsymbol{x}_1$,...). Rasmussen leaves the question of the relations of $\eta(\boldsymbol{x}_o,\boldsymbol{x}_1)$ and $\psi(\boldsymbol{x}_o,\boldsymbol{x}_1)$ to measured records of $\eta(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ somewhat open.

To obtain (2.1) and (2.5) from (A.1) and (A.2), one should adopt the following definition:

$$\eta(\boldsymbol{x}_o,\boldsymbol{x}_1)\equiv\sum_{m,n=-\infty}^{\infty}c_{m,n}(\boldsymbol{x}_1)e^{i\pi(mx_o+ny_o)/L} \tag{A.3}$$

where

$$c_{m,n}(\boldsymbol{x}_1)=\frac{1}{4L^2}\int_{x_1+L}^{x_1+L}\int_{y_1+L}^{y_1+L}\eta(\boldsymbol{x})e^{-i\pi(mx+ny)/L}dxdy \tag{A.4}$$

and similarly for $\psi(\boldsymbol{x}_o,\boldsymbol{x}_1)$.

Many other definitions of η $(\boldsymbol{x}_o,\boldsymbol{x}_1)$ are possible. One of them is:

$$\eta(\boldsymbol{x}_o,\boldsymbol{x}_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}c(\boldsymbol{k},\boldsymbol{x}_1)e^{i\boldsymbol{k}\cdot\boldsymbol{x}_o}d\boldsymbol{k} \tag{A.5}$$

where

$$c(\boldsymbol{k},\boldsymbol{x}_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta(\boldsymbol{x})H(\boldsymbol{x},\boldsymbol{x}_1,L)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}d\boldsymbol{x} \tag{A.6}$$

and

$$H=\begin{cases}1, \text{ when} |\mathrm{x}-\mathrm{x}_1| \text{ and } |\mathrm{y}-\mathrm{y}_1| \text{ are both } < \mathrm{L}\\ 0, \text{otherwise}\end{cases} \tag{A.7}$$

In contrast to (A.3) and (A.4), definitions (A.5), (A.6) and (A.7) do not lead to a discrete presentation. Note that all equations in this paper (excluding the examples in sections 5 and 6) can be easily rewritten for continuous cases.

Appendix B: The relationship between $\rho_{m,n}$ and the wavenumber spectrum.

Note that

$$\rho_{M,N} = < B_{M,N}(\boldsymbol{x}_1) B^*_{M,N}(\boldsymbol{x}_2) > \tag{B.1}$$

where

$$B_{M,N}(\boldsymbol{x}) = \frac{\pi}{2L^2} \int_{-L}^{L} [(\frac{g}{2\omega_{M,N}})^{1/2} \eta(\boldsymbol{x}+\xi) + i(\frac{\omega_{M,N}}{2g})^{1/2} \psi(\boldsymbol{x}+\xi)] e^{-i[\boldsymbol{k}_{M,N}\cdot(\boldsymbol{x}+\xi) - \omega_{M,N} t]} d\xi$$

Substituting (B.2) into (B.1) yields (B.2)

$$\rho_{M,N} = \frac{\pi^2}{4L^4} \int_{-L}^{+L} \int_{-L}^{+L} \left\{ \frac{g}{2\omega_{M,N}} < \eta(x_1+\xi_1)\eta(x_2+\xi_2) > - \frac{i}{2} < \eta(x_1+\xi_1)\psi(x_2+\xi_2) - \right.$$

$$\left. - \psi(x_1+\xi_1)\eta(x_2+\xi_2) > + \frac{\omega_{M,N}}{2g} < \psi(x_1+\xi_1)\psi(x_2+\xi_2) > \right\} \cdot e^{-ik_{M,N}\cdot(x_1 - x_2 + \xi_1 - \xi_2)} d\xi_1 \, d\xi_2 \tag{B.3}$$

For a homogeneous ocean $\rho_{M,N}$ is a function of $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. Taking the $\boldsymbol{r}$ to $\boldsymbol{p}$ Fourier transform of (B.3):

$$\hat{\rho}^{\,h}_{M,N}(\boldsymbol{p}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho^{(h)}_{M,N}(\boldsymbol{r}) e^{-i\boldsymbol{p}.\boldsymbol{r}} d\boldsymbol{r} =$$

$$= \frac{8\pi^3}{L^4} \frac{\sin^2(p_x L)}{p_x^2} \frac{\sin^2(p_y L}{p_y^2}). \tag{B.4}$$

$$.\{\frac{g}{2\omega_{M,N}} S_{\eta\eta}(\boldsymbol{p}+\boldsymbol{k}_{M,N}) - \frac{i}{2}[S_{\eta\psi}(\boldsymbol{p}+\boldsymbol{k}_{M,N}) - S_{\psi\eta}(\boldsymbol{p}+\boldsymbol{k}_{M,N})] + \frac{\omega_{M,N}}{2g} S_{\psi\psi}(\boldsymbol{p}+\boldsymbol{k}_{M,N})\}$$

where

$$S_{\eta\psi}(\boldsymbol{k}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} < \eta(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{r}) > e^{-i\boldsymbol{k}.\boldsymbol{r}} d\boldsymbol{r} \tag{B.5}$$

According to linear theory, $S_{\psi\psi}(\boldsymbol{k}) = \frac{g}{\omega^2} S_{\eta\eta}(\boldsymbol{k})$ and $S_{\psi\eta} = S_{\eta\psi} = 0$, so that (A.4) reduces to

$$\hat{\rho}^{(h)}_{M,N}(\boldsymbol{p}) = \frac{4\pi^3 g}{L^4 \omega_{M,N}} \frac{\sin^2(p_x L)}{p_x^2} \frac{\sin^2(p_y L)}{p_y^2} [1 + \frac{\omega^2_{M,N}}{\omega^2(\boldsymbol{k}_{M,N}+\boldsymbol{p})}] S_{\eta\eta}(\boldsymbol{k}_{M,N}+\boldsymbol{p}), \tag{B.6}$$

where $S_{\eta\eta}$ is nothing but the wavenumber spectrum.