An overview of recent developments in numerical methods for atmospheric modelling

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Abstract

Since 1991, when the previous Seminar on Numerical Methods was held, there has been significant progress in a number of aspects of this topic. This overview highlights some of the successes as well as some promising areas which have yet to mature.

1. INTRODUCTION

The aim of this paper is to provide a brief survey of recent developments in numerical methods for atmospheric modelling, and in particular for numerical weather prediction, in order to set the scene for subsequent papers in these Seminar Proceedings. For this purpose, it is convenient to define "recent" as referring (roughly speaking) to the period since the last ECMWF Seminar on Numerical Methods, held in 1991.

The choice of topics covered will be a somewhat personal one, reflecting the areas in which the author has either been directly involved or taken a close interest. The most glaring omission is perhaps the continuing development of nonhydrostatic models (but see *Bubnová,* 1999; *Laprise*, 1999; *Smolarkiewicz et al.*, 1999). To complete the overview, an attempt will be made to provide an "end-of-term report" comparing the rates of progress in different aspects of the field of numerical methods for atmospheric modelling.

2. DEVELOPMENTS AT ECMWF

Given finite resources, there are two ways to improve the accuracy of a numerical model. The first is simply to find a more accurate algorithm. The second is to develop an algorithm which obtains the same accuracy but more cheaply, and then to increase the spatial resolution (and check that this does indeed bring higher accuracy).

Recent developments at ECMWF provide a striking example of the second approach. At the time of the last Seminar on Numerical Methods, the operational model was an Eulerian spectral model with triangular truncation T106 and 19 levels, running on a "quadratic" (320x160) Gaussian grid. Seven years later at the time of the present Seminar, the horizontal resolution has been increased to T319 and the vertical resolution to 31 levels. Using the 1991 numerical scheme, the current model would be roughly 50 times more expensive to run than its 1991 counterpart. In fact, the cost is *similar.*

This enormous improvement in efficiency is the product of a number of steps: the use of a “reduced” Gaussian grid to eliminate redundant computations as the poles are approached (*Hortal and Simmons*, 1991), the implementation of a three-time-level semi-Lagrangian scheme (*Ritchie et al.*, 1995), the subsequent conversion to a two-time-level scheme (*Temperton et al.*, 1999) and the use of a “linear” Gaussian grid together with some improvements to the two-time-level scheme (*Hortal* 1999b). These developments are described in much more detail in these Proceedings by *Hortal* (1999a), who argues that with the operational introduction early in 1999 of a 50-level model extending into the stratosphere, the improvement in efficiency over the 1991 numerical scheme increases further to a factor of about 72. This gain has been of paramount importance in improving the overall capability of the operational assimilation/forecast system.

3. MATURING OF SEMI-LAGRANGIAN SCHEMES

Staniforth and Côté (1991) provided a comprehensive review of the development of semi-Lagrangian methods up to that time. Since then, there has been progress in a number of areas which they identified as being ripe for further development.

Two-time-level schemes, which in principle are twice as efficient as their three-time-level counterparts, are increasingly being used in operational three-dimensional forecast models. *McDonald and Haugen* (1992, 1993) developed a two-time-level finite-difference limited-area model. Applications to global finite-difference medium-range forecast models have been described by *Chen and Bates* (1996) and *Moorthi* (1997). *Côté et al.* (1998a, 1998b) present a global finite-element model based on a two-time-level semi-Lagrangian scheme. The application to a global spectral model at ECMWF is described by *Temperton et al.* (1999) and *Hortal* (1999a, 1999b). The papers by Hortal also develop the concept of coupling a semi-Lagrangian scheme in a spectral model with a *linear* grid, an idea first proposed by *Côté and Staniforth* (1988). Thus, the currently operational T319 model at ECMWF uses the same grid in physical space as did the previously operational T213 model (*Ritchie et al.*, 1995).

Conventional semi-Lagrangian schemes have no formal conservation properties. In practice, the drift in the global mean surface pressure during the course of a 10-day forecast has been found to be negligible. However, conservation of mass (and of advected quantities in general) is an issue in climate models and seasonal forecasting applications. *Priestley* (1993) adapted the quasi-monotone semi-Lagrangian scheme of *Bermejo and Staniforth* (1992) to make it conservative for passively advected fields. *Gravel and Staniforth* (1994) then generalized Priestley’s algorithm to ensure conservation of mass in a semi-Lagrangian scheme for the shallow-water equations. Taking a different approach, *Leslie and Purser* (1995) showed that conservation of mass and passive tracers could be achieved using a modified form of cascade interpolation (see below).

Modelling flow over mountains with a semi-Lagrangian integration scheme can lead to problems in the form of a spurious resonant response to steady orographic forcing. The mechanism was clarified by *Rivest et al.* (1994). *Ritchie and Tanguay* (1996) proposed a modification to the semi-Lagrangian scheme which alleviates the problem, and it has been widely adopted. In the ECMWF model it was found that, besides improving the flow over mountains in the forecast, this modification also improved the mass conservation properties of the semi-Lagrangian scheme (*Temperton et al.*, 1999).

While the use of semi-Lagrangian schemes in short- and medium-range numerical weather predicion models has become general, climate modellers have been more reluctant to accept them. At low resolutions the efficiency advantage of semi-Lagrangian schemes is less marked, since the timestep is often limited by the physical parametrizations rather than the advective stability criterion. Also, there have been worries about conservation properties and instances of different climatologies being produced by semi-Lagrangian scchemes compared with corresponding Eulerian models. However, *Williamson and Olson* (1998) point out that Eulerian simulations can be "right for the wrong reason", and that the more accurate semi-Lagrangian treatment of vertical advection is a particular advantage. They conclude that "there is no longer any reason not to use semi-Lagrangian approximations as the basis of global atmospheric climate models". See also *Williamson et al.* (1998). Furthermore, in the context of spectral climate models, *Williamson* (1997) notes that the use of a linear grid in conjunction with semi-Lagrangian advection provides higher resolution at very little extra cost: the quality of the simulation depends on the spectral resolution rather than on the resolution in physical space.

Semi-Lagrangian schemes always require interpolation of quantities at the departure points; since these interpolations have to be of at least cubic accuracy for the advected variables, this represents a fairly expensive part of the algorithm in a three-dimensional model. *Purser and Leslie* (1991) showed that the usual three-dimensional interpolation at each departure point could be replaced by a "cascade" of one-dimensional interpolations, giving a substantial economy which could be reinvested in higher-order approximations. *Nair et al.* (1999a) have simplified the procedure and made it more efficient. They have also shown (*Nair et al.*, 1999b) that it can be made to work on the sphere, provided that the grid is of "tensor product" latitude-longitude form. Unfortunately, the applicability to a reduced grid as used at ECMWF has not yet been demonstrated.

Semi-Lagrangian schemes have thus matured considerably during the 1990s. It is natural to ask whether they will continue to be so useful as numerical weather prediction models move to higher and higher resolution. Based upon assumptions about the shape of the atmospheric energy spectrum, *Bartello and Thomas* (1996) claim that the advantages of semi-Lagrangian schemes will disappear at high resolution. However, *Côté et al.* (1998a) argue that the analysis by Bartello and Thomas is unconvincing. Time will tell who is right.

4. VARIABLE RESOLUTION

Both in numerical weather prediction and in climate simulation, attention is often focussed on a particular region of the globe. Since models on limited domains inevitably suffer from the need to impose artificial lateral boundary conditions, a neat solution is to run a variable-resolution global model in which the resolution is highest over the region of interest. During the 1990s two contrasting approaches to implementing this idea have been pursued, with approximately equal success. Both start by rotating the spherical coordinate system, but thereafter they diverge.

In the first approach, particularly suited to finite-difference or finite-element horizontal discretizations, the coordinate system is rotated so that the region of interest lies on the equator of the "computational" sphere. A latitude-longitude grid is then imposed, with resolution concentrated over the region of interest. *Côté et al.* (1993) demonstrated the feasibility of this approach for the shallow-water equations, and their work formed the basis of the new Canadian operational model (*Côté et al.* 1998a, 1998b). Meanwhile, *Fox-Rabinovitz et al.* (1997) have demonstrated that a similar strategy can be employed in a finite-difference climate model.

In the second approach, particularly suited to a spectral horizontal discretization, the coordinate system is rotated to bring the computational (north) pole to the centre of the region of interest, and a coordinate stretching is then applied in the new "north-south" direction. This introduces map factors into the governing equations, after which a "uniform" resolution (or spectral discretization) can be applied on the transformed sphere. This approach was demonstrated using the shallow-water equations by *Courtier and Geleyn* (1988), and is now incorporated in the operational model of Météo-France. Important modifications to the original semi-implicit and diffusion operators are described by *Yessad and Bénard* (1996). In principle, the coordinate stretching can provide arbitrarily high resolution at the computational pole; however, *Caian and Geleyn* (1997) demonstrate that there appear to be limitations in practice.

5. ALTERNATIVE GRIDS ON THE SPHERE

In a global spectral model based on spherical harmonics, triangular truncation provides the equivalent of isotropic uniform resolution. With the use of a reduced Gaussian grid (*Hortal and Simmons,* 1991), the property of uniform resolution is approximately preserved in gridpoint space. It is notoriously difficult to achieve this desirable property using a finite-difference/finite-element discretization. A promising approach which has received much attention in the past few years is to subdivide the surface of the globe into spherical triangles, starting from an icosahedron inscribed within the sphere. This idea was first tried in the late 1960s (*Williamson* 1968;

Sadourny et al. 1968). After languishing for many years, it has recently enjoyed a considerable revival (see for example *Heikes and Randall* 1995a, 1995b). The forthcoming operational model of the Deutscher Wetterdienst is in fact based on such an icosahedral grid (*Majewski*, 1999). Another possibility currently being pursued is to project the surface of the sphere onto a cube (*McGregor* 1996), again a revival of an old idea (*Sadourny* 1972).

6. POTENTIAL VORTICITY CONSERVATION

The landmark paper by *Hoskins et al.* (1985) established (or re-established) the central importance of potential vorticity as a dynamical variable. It is thus natural to consider using potential vorticity as a basic prognostic variable in a primitive-equation model. *Bates et al.* (1995) showed the feasibility of this idea in a global semi-Lagrangian shallow-water model. Some advantages over the conventional formulation were demonstrated by *Li and Bates* (1996). This is a promising start; the extension to a three-dimensional model is still eagerly awaited.

7. VERTICAL DISCRETIZATION

In most models (including that of ECMWF), the vertical discretization is the most "embarrassing" part of the numerical formulation. In the case of the ECMWF model the vertical discretization would only be second-order accurate on a uniform grid, and since the grid actually used is non-uniform the accuracy is degraded to first-order. The situation has been redeemed to some extent by the introduction of the semi-Lagrangian scheme, which provides fourth-order accuracy for the vertical advection even on a non-uniform grid.

There seems to have been little progress recently in this area. *Leslie and Purser* (1991) advocated the use of high-order quadrature formulae in the vertical discretization, and demonstrated a positive impact on forecast quality in a regional model. Taking an example from outside the field of numerical weather prediction, *Spotz and Carey* (1998) have shown how to construct high-order compact difference schemes on a non-uniform grid: this might be a fruitful line of research.

A related question concerns the placement or staggering of variables in the vertical. Most centres (including ECMWF) use the so-called Lorenz grid. However, the new dynamical scheme being developed at the UK Meteorological Office uses the alternative Charney-Phillips grid, which does have some potential advantages (*Davies et al.*, 1999).

8. TEST CASES AND DYNAMICAL CORES

Testing a new numerical scheme in a three-dimensional model presents difficulties because of the lack of test problems with known solutions. For horizontal discretizations it is common practice to start by testing the scheme in the shallow-water equations, which are simpler but contain most of the essential features. Several recent examples have already been cited. A standard set of test problems for the shallow-water equations on the sphere was proposed by *Williamson et al.* (1992), and has been widely adopted.

In three-dimensional models, a further difficulty in testing numerical schemes for the dynamical equations lies in disentangling the behaviour of the dynamics from that of the physics. To overcome this problem, *Held and Suarez* (1994) proposed a "dynamical core" framework in which the physical parametrizations are replaced by a simple Newtonian relaxation of the temperature field towards a specified zonally symmetric state, and Rayleigh damping of low-level winds to represent boundary-layer friction. Numerical schemes for the dynamics, and for example their behaviour as a function of resolution, can then be assessed by running long integrations and comparing the resulting "climates". The Held-Suarez test and some variants (*Boer and Denis* 1997; *Williamson et al.* 1998) have become a valuable and popular tool in the development of numerical methods for global atmospheric models.

9. TOWARDS VERY HIGH RESOLUTION: THE FUTURE OF SPECTRAL METHODS ON THE SPHERE

It has often been argued (especially by finite-difference/finite-element modellers) that in the long run, as horizontal resolution becomes higher and higher, global spectral models are doomed to extinction. There are two main strands to the argument.

The first is that spectral models require global communication (but so do gridpoint models if the time integration algorithm is semi-implicit), and that they will not run efficiently on massively parallel computers. However, experience at ECMWF (and elsewhere) suggests that a simple strategy of transposing the data structures when required is all that is needed for models to run efficiently on machines with up to at least $O(10^3)$ processors (*Dent* 1999).

The second is that the cost of the Legendre transforms will eventually swamp the calculations, since for a model truncated at wavenumber N this cost (per timestep) grows like N^3 while other costs grow like N^2 or at worst $N^2 \log N$. The highest resolution so far run at ECMWF has been T639, at which point the Legendre transforms took about 20% of the total CPU time. The point at which the spectral technique ceases to be viable is thus some way off, but it can be seen on the horizon.

The future of spectral methods on the sphere would be assured if we could find a robust and efficient fast Legendre transform analogous to the fast Fourier transform, but so far this has proved elusive. A possible way around the problem stems from the observation that if a field on the sphere can be represented as a truncated spherical harmonic series, then it can also be represented as a similarly truncated double Fourier series (the converse is not true). It is therefore possible to base a model on double Fourier series, the Fourier coefficients (or the corresponding fields in physical space) being filtered so that they remain in the subspace spanned by the corresponding spherical harmonic basis. This approach has recently been explored by *Spotz et al.* (1998). The remaining problem is to find a "fast" way of doing the filtering. A very promising candidate has been proposed by *Jakob-Chien and Alpert* (1997); see also *Yarvin and Rokhlin* (1998). However, we have yet to see an actual model based on a fast algorithm of this type.

10 . SUMMARY

There has been significant progress during the last few years on a number of fronts in the field of numerical methods for atmospheric modelling. The only area identified here as disappointing in terms of recent progress is that of vertical discretization.

Promising beginnings have been made on models based on the conservation of potential vorticity and on circumventing the "Legendre transform barrier" in global spectral models, but these topics have yet to mature.

Semi-Lagrangian schemes have matured considerably; they are now in widespread use for short- and medium-range forecast models, and are gradually being accepted in climate models. Techniques for achieving variable resolution over the sphere have now been incorporated in operational models, as have alternative grids based on the icosahedron. Testing new methods has become easier thanks to the use of standard test cases and the dynamical core framework. Global spectral models have been run at resolutions as high as T639. In summary, the period since the last Seminar on Numerical Methods has been one of exciting advances.

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