Systematic and Random Error Budgets

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1. Introduction

Six cases arise when considering models which are perfect or imperfect and initial conditions which have no, purely random or a mixture of systematic and random error. Of these six cases, only two result in "pure" forms of error in the forecast. The "deterministic" case of a perfect model and perfect initial conditions has no error but is not of much interest. The "classical predictability" case of a perfect model and purely random initial error gives forecast error which is purely random. All other cases, including the practical problem of weather forecasting, result in a mixture of systematic and random error in the forecast. This short note discusses some of the results of a preliminary analysis of the budgets of growth and interaction of systematic and random error in an extended range forecasting experiment. A more complete description of the experiment and of the derivation and evaluation of the budget equations is given in Boer (1992).

2. Data

The data used are from a dynamical extended range forecast experiment using the Canadian Climate Centre T20L10 low resolution general circulation model (Boer et al. 1984a,b). Monthly forecasts are performed for each of the eight Januaries from 1979 to 1986. Up to six individual forecasts are made for each month. The 6 observing periods (3 days) prior to the first day of the month provide the initial conditions. NMC global analyses are used both for initial conditions and as verifying data. No initialization is performed. Of the 48 possible forecasts (6 cases for each of the eight years), 42 are available for analysis.

The boundary conditions are fixed in an "operational" way using sea surface temperature and snow line information available prior to the beginning of the forecast. The observed sea surface temperature anomaly, which is kept constant during the integration, is added to the time varying climatological sea surface temperatures of the model. The snow amount evolves from its initial value as the forecast progresses.

3. January mean 500mb height

Results for the forecasting of January mean 500mb height may be summarized as follows: (1) There is marginal forecast skill for the 30-90N Northern Hemisphere region as measured by the anomaly correlation coefficient; (2) Although the systematic error is not large, when it is removed the skill level for this region increases slightly so that half of the years display anomaly correlation values greater than 0.5 for the ensemble mean forecast; (3) For this same region there is a statistically significant relation between ensemble mean forecast skill and the spread between the individual forecasts; (4) For the North American region from 30-70N and 50-140W, the anomaly correlation coefficients display more extreme values and more sensitivity to the removal of systematic error. After removal of the error some 7 of the 8 years display anomaly correlations for the ensemble mean forecast of near or greater than 0.5; (5) The relationship between forecast skill and spread for the North American region is not statistically significant; (6) There exist geographically connected regions for which the temporal anomaly correlation (as opposed to the spatially averaged values discussed above) show modestly high values but there is no obvious relationship between these regions and parameters of the flow. These results are discussed in more detail in Boer(1992).

4. Error equations

In keeping with the use of the 500mb height field as the traditional forecast verification variable, error budget equations are developed, interpreted and evaluated based on the information contained in that field.

a. The equation of motion at 500mb

At 500mb the rotational component dominates the flow everywhere except possibly at very low latitudes and the vorticity equation, which governs this component of the flow, is basic to middle latitude dynamics. The 500mb height is used directly by decomposing the velocity term into a mid-latitude geostrophic component and the remaining ageostrophic term and expanding the coriolis parameter f about some middle latitude, $\varphi_* = 45N$, to give the equation for the geostrophic velocity which is consistent with the vorticity equation as

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{\partial f}{\partial \varphi} \Big|_* (\varphi - \varphi_*)\ \mathbf{k} \times \mathbf{V} = \mathbf{R} \tag{1}$$

where $\mathbf{V}$ is the geostrophic velocity $\mathbf{V} = f_*^{-1}\ \mathbf{k} \times \nabla\Phi$. The non-linear geostrophic and barotropic processes which affect the evolution of the error at 500mb are represented on the left-hand side of the equation while all remaining ageostrophic, baroclinic and dissipative terms are represented on the right-hand side by $\mathbf{R}$.

b. The error equations

The velocities in (1) are relabeled as

$$\mathbf{V}_f = \mathbf{V}_o + \mathbf{V}$$

where unsubscripted variables now represent the error while the forecast and "observed" values (actually objectively analyzed values) are indicated by the appropriate subscripts.

The mean square error in velocity is a function of forecast time and is referred to as the error kinetic energy. It is defined as

$$k(t) = \frac{1}{2}\,\overline{\mathbf{V} \cdot \mathbf{V}}$$

where the overbar indicates an "ensemble average" over many forecasts at the same forecast time. The equation governing the growth of this mean square error is obtained in the usual way by subtracting observed from the forecast version of (1), multiplying by $\mathbf{V}\cdot$, taking the ensemble average indicated by an overbar and rearranging terms to give

$$\frac{\partial k}{\partial t} = -\nabla \cdot \left\{ \overline{\frac{\mathbf{V}\cdot\mathbf{V}}{2}\ \mathbf{V}_f} \right\} - \left\{ \overline{u\mathbf{V}\cdot\nabla u_o} + \overline{v\mathbf{V}\cdot\nabla v_o} \right\} + \overline{\mathbf{V}\cdot\mathbf{R}} \tag{2}$$

The equation states that the growth of mean square error in velocity at a point is governed by the convergence or divergence of the flux of such error by the forecast flow, by a generation term representing non-linear barotropic interactions between the errors and the correct flow and by the remaining source/sink term involving the covariance of the erroneous velocity with **R** which represents errors in non-geostrophic, baroclinic and dissipative terms.

c. Systematic and random error

Systematic error is defined as that part of the error that survives ensemble averaging over many forecasts and the random error as that remaining. The error is decomposed into these two components as

$$\mathbf{V}(t) = \overline{\mathbf{V}}(t) + \mathbf{V}'(t) = \mathbf{V}_s(t) + \mathbf{V}'(t)$$

where the subscript "s" is also used to indicate the systematic component of the error while the random error, for which $\overline{\mathbf{V}'} = 0$, is indicated by a prime.

Total, systematic and random error components are given by

$$k\,(t) = k_S(t) + k_R(t) = \frac{1}{2}\,\overline{\mathbf{V}.\mathbf{V}}$$

$$k_S(t) = \frac{1}{2}\,\mathbf{V}_S.\mathbf{V}_S$$

$$k_R(t) = \frac{1}{2}\,\overline{\mathbf{V}'.\mathbf{V}'}\ .$$

d. Systematic and random error equations

Systematic and random error equations are obtained by respectively multiplying (1) by $\mathbf{V}_S\cdot$ and $\mathbf{V}'\cdot$ taking the ensemble average and rearranging the terms. Although terms in these equations may be evaluated locally they are not done so here. Rather, averages over the Northern Hemisphere, represented by <..>, are considered. The resulting equations may be written in several forms as discussed in Boer (1992). Here we consider the form

$$\frac{\partial K_S}{\partial t} = G_{SS} + CG_{SR} + S_S$$

$$\frac{\partial K_R}{\partial t} = G_{RR} + CG_{RS} + S_R \ . \qquad (3)$$

The rate of change of systematic and random error is a consequence of: (1) "pure" non-linear barotropic generation terms

$$G_{SS} = - \langle u_S \mathbf{V}_S \cdot \nabla \bar{u}_o + v_S \mathbf{V}_S \cdot \nabla \bar{v}_o \rangle$$

$$G_{RR} = - \langle \overline{u' \mathbf{V}' \cdot \nabla u_o} + \overline{v' \mathbf{V}' \cdot \nabla v_o} \rangle$$

which contain only systematic or random components of the error but not both; (2) "mixed" generation/conversion terms

$$CG_{SR} = \langle (\overline{u' \mathbf{V}_o} + \overline{u' \mathbf{V}'}) \cdot \nabla u_S + (\overline{v' \mathbf{V}_o} + \overline{v' \mathbf{V}'}) \cdot \nabla v_S - u_S \overline{\mathbf{V}' \cdot \nabla u_o} - v_S \overline{\mathbf{V}' \cdot \nabla v_o} \rangle$$

$$CG_{RS} = - \langle (\overline{u' \mathbf{V}_o} + \overline{u' \mathbf{V}'}) \cdot \nabla u_S + (\overline{v' \mathbf{V}_o} + \overline{v' \mathbf{V}'}) \cdot \nabla v_S - \overline{u' \mathbf{V}_S \cdot \nabla u_o} - \overline{v' \mathbf{V}_S \cdot \nabla v_o} \rangle$$

which depend on the existence of both systematic and random forms of error and which are zero in the absence of either; and (3) the remaining source/sink terms

$$S_S = \langle \mathbf{V}_S \cdot \mathbf{R}_S \rangle$$

$$S_R = \langle \overline{\mathbf{V}' \cdot \mathbf{R}'} \rangle$$

representing all remaining processes including baroclinic and non-ageostrophic effects. The terms in these equations are evaluated directly from data with the exception of the source/sink terms which are obtained as residuals.

Since this version of the equations isolates the terms representing the interaction between the systematic and random error it gives some information on the possibility of "correcting" forecasts by "subtracting out" systematic error after the fact (e.g. Miyakoda et al., 1986, Tracton et al., 1989, Déqué, 1991) or by providing a "corrective" forcing term on the right hand side of the equation (e.g. Sausen and Ponater, 1990).

5. Evaluation of terms in the budgets

The error budgets are in differentiated form where the rate of change of the error is given in terms of the rates of generation and conversion and of the source/sink term. Quantities may be presented in terms of these rates but it is often easier to integrate the equations in time and to consider integrated "contributions" to the budgets and this is done here. Integrating equations (3) over forecast time gives

$$K_R = \tilde{G}_{RR} + \tilde{C}\tilde{G}_{RS} + \tilde{S}_R$$

$$K_S = \tilde{G}_{SS} + \tilde{C}\tilde{G}_{SR} + \tilde{S}_S$$

where the time integration is indicated by a tilde as $\tilde{X}(t) = \int_o^t X(\tau)\, d\tau$. The terms in the equations are displayed in "additive" form (where the distances between the curves represent the magnitudes of the quantities) in Figures (1) and (2).

a. The random error budget

The picture is relatively straightforward as shown in Figure (1). Not surprisingly, at the earliest forecast times the growth of random error is due mainly to the error source/sink term rather than to the pure random non-linear baroclinic generation term. Since the error is initially small the non-linear generation term is small. As the forecast progresses, and as the error becomes somewhat larger, the non-linear generation term becomes increasingly important and comes to dominate the budget.

Throughout the forecast, the contribution to the random error from the interaction with the systematic error via the mixed term $\tilde{C}\tilde{G}_{RS}$ is comparatively small. This lack of a strong non-linear interaction suggests that it may be possible to "subtract-out" the systematic error after the fact.

At early forecast times the error source/sink term, representing errors other than those accounted for by the non-linear barotropic generation term, is the main source of error. Clearly the reduction of this

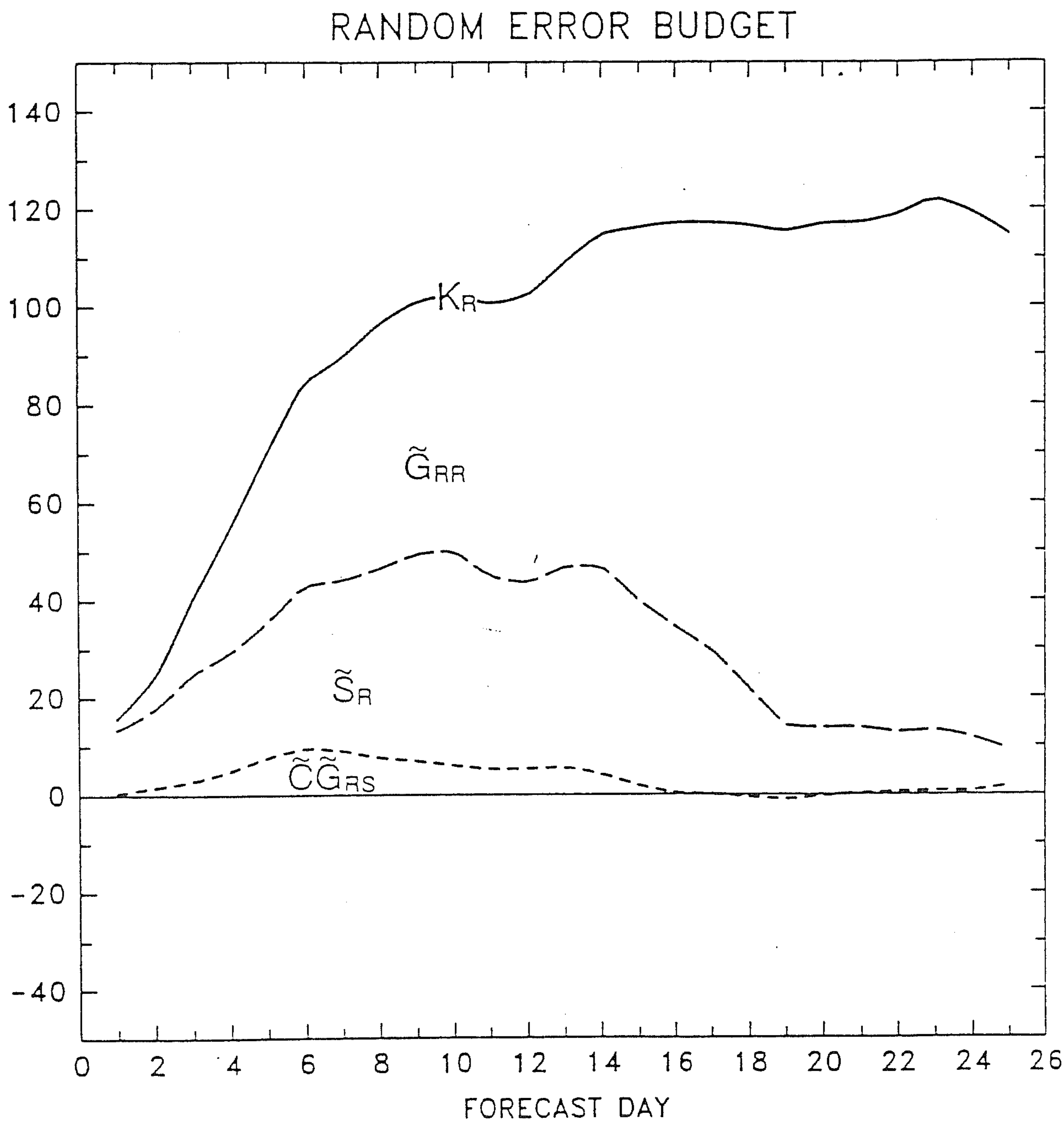

Figure 1. The integrated random error budget $K_R = \tilde{G}_{RR} + \tilde{C}\tilde{G}_{RS} + \tilde{S}_R$ in additive form. The distance from the axis to the dotted curve is the non-linear generation/conversion term $\tilde{C}\tilde{G}_{RS}$, the distance from the dotted to the dashed curve baroclinic source/sink term S_R and the distance from the dashed to the solid curve the pure random non-linear barotropic generation term $\tilde{G}_{RR}$. They add to give the random error variance K_R represented by the solid curve. Units m^2s^{-2}.

source of error would reduce error growth directly as well as delaying the generation of error by the non-linear generation term.

Although it is not clear how this reduction can be accomplished, it is plausible that improved parameterization of the physical processes in the model might be more effective in reducing error growth at early forecast times by reducing S than an increase in resolution which might give some improvement in G. One aspect of this could be tested by comparing budgets for a model at different resolutions.

b. The systematic error budget

Terms in the time-integrated version of the systematic error budget are displayed in Figure (2). Note that the scale of the diagram differs from that of the random error budget by a factor of five. As was the case for the random error, the source/sink term initiates the error growth at the earliest forecast times but the contribution does not grow dramatically thereafter. The non-linear interaction between the systematic and the much larger random error is an important contributor to systematic error growth at early forecast times. This in turn suggests that it might be difficult to reduce the systematic error by a corrective forcing term on the right hand side of equation (1).

The pure non-linear generation of systematic error is small for the first four days of the forecast and then begins to grow until it dominates the budget. The systematic error apparently saturates more or less at the same time as the random error. Presumably the delay in the growth of error due to the pure non-linear generation term is a consequence of the smallness of the systematic error component at early forecast times as well as the rather small and more or less constant source/sink contribution. The implication is that the systematic error would grow considerably more slowly if it did not interact with the random error at early forecast times. Since this interaction depends on the size of both forms of error, reducing either should have a feedback effect in decreasing systematic error.

6. Concluding comments

An extended range forecasting experiment based on the 8 Januaries from 1979 to 1986 is analyzed in terms of the "traditional" extended range

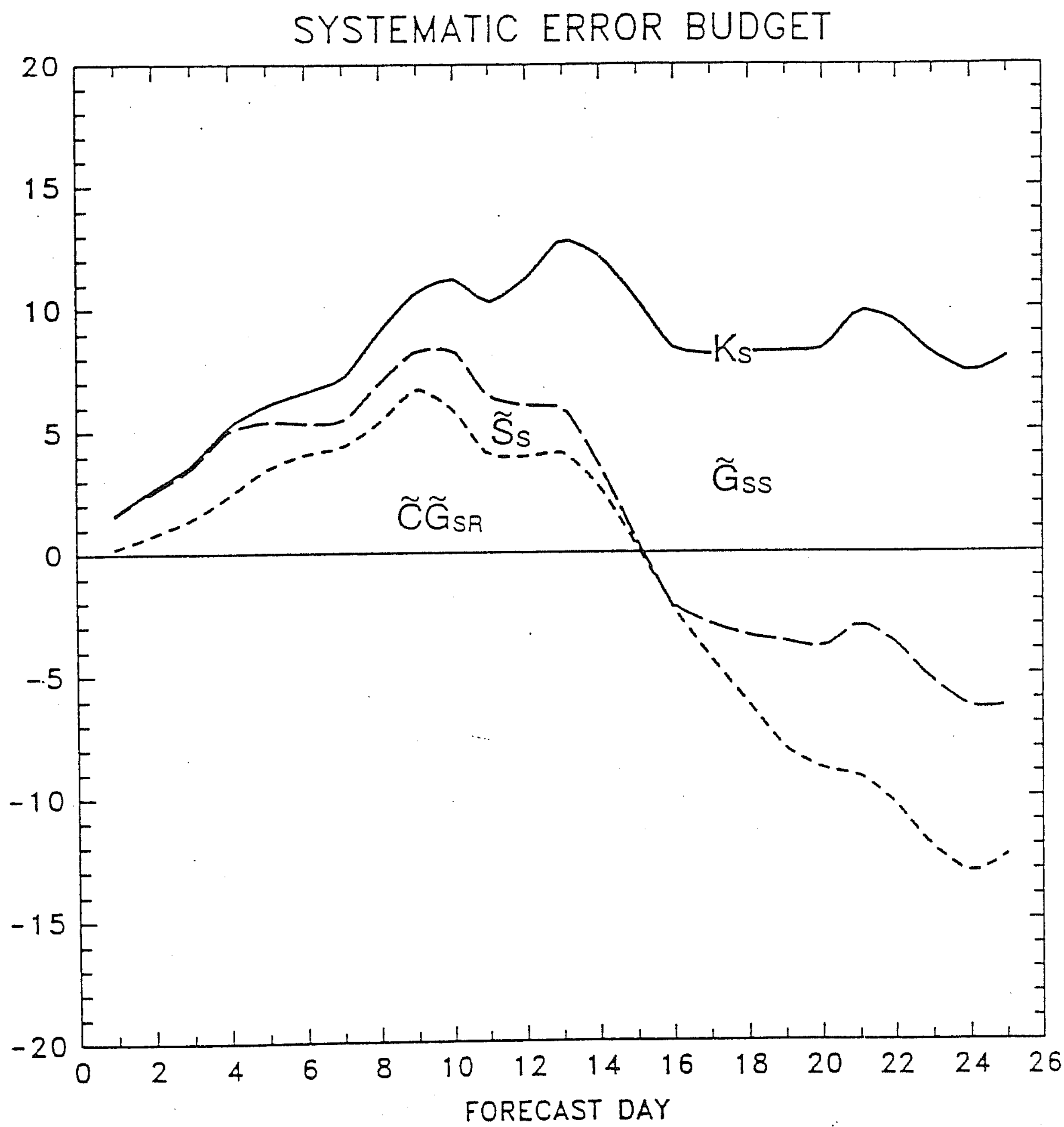

Figure 2. The integrated systematic error budget $K_S = \tilde{G}_{SS} + \tilde{C}\tilde{G}_{SR} + \tilde{S}_S$ in additive form. The distance from the axis to the dotted curve is the non-linear generation/conversion term $\tilde{C}\tilde{G}_{SR}$, the distance from the dotted to the dashed curve baroclinic source/sink term S_R and the distance from the dashed to the solid curve the pure random non-linear barotropic generation term $\tilde{G}_{SS}$. They add to give the random error variance K_S represented by the solid curve. Units m^2s^{-2}.

forecast variable, namely the Northern Hemisphere 500mb height field. Budget equations for systematic and random error are developed and applied in their "integrated" form. The budgets of the two components of the error are rather different. From the point of view of the random error budget the interaction with the systematic error via the mixed non-linear conversion term is not very important. This gives some hope that the systematic error can be "subtracted out" in this (but perhaps not every) case.

At early forecast times random error grows mainly due to the source/sink term representing baroclinic and other components of the flow including errors in forcing. At later forecast times the barotropic non-linear error generation term dominates the budget. If this behavior is general it implies that improvement to the source/sink term may delay the important non-linear error growth.

The systematic error variance is only about 10% of that of the random error. It's growth is importantly affected by its interaction with the random error via the non-linear mixed term. While all terms in the budget are ultimately important, this non-linear interaction with the random error is especially important at early forecast times.

The application of these equations to the results of other models can suggest if they are acting similarly in terms of the growth and generation of random and systematic error. Their application to results from the same model at different resolutions should show how the different components of the error budget depend on resolution by itself.

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