PROPERTIES OF THE PARTIAL DIFFERENTIAL EQUATIONS GOVERNING VARIOUS TYPES OF ATMOSPHERIC MOTIONS AND IMPLICATIONS FOR NUMERICAL METHODS

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Summary: This paper reviews known results on the behaviour of the solutions to both the partial differential equations governing the complete motion of the atmosphere and various reduced versions of the equations governing motions of direct relevance to weather forecasting. These results usually depend on certain special conservation properties of the equations which must be respected by numerical methods.

1. INTRODUCTION

It has been recognised for a long time that successful numerical weather prediction requires careful design of the numerical methods used to approximate the governing equations. The basic requirements for consistency and stability, together implying convergence, are set out in Richtmyer and Morton (1967). While it is usually easy to check consistency, proofs of stability of an approximation to a set of nonlinear equations are not often possible. The pioneering work of Arakawa (1966) suggested that exact preservation of integral invariants could aid nonlinear stability, and this approach has proved fruitful ever since.

Further progress in identifying stable and accurate methods requires a greater understanding of the behaviour of the governing equations. No proof of nonlinear stability is possible without proving that the problem to be solved has a bounded solution. Similarly, a formally high order of accuracy does not result in the same high rate of convergence except to a solution which is sufficiently smooth. There has been a great deal of progress in understanding these issues in recent years.

It is not sufficient to consider the behaviour of the numerical methods used in forecast models as methods for solving the full compressible Navier-Stokes equations, even though these are the equations used. It is also essential that the models are capable of exactly representing important lower order balances, such as a state of rest in hydrostatic balance. This paper therefore also reviews results on a range of balanced models which describe most of the weather producing motions. It is necessary that forecast models contain sufficiently accurate 'hidden' approximations to the balanced motion.

The potential subject matter is very large, and this paper therefore makes a selection of those topics which currently appear most important. Section 2 deals with the implications of Lagrangian conservation properties in incompressible flows. These properties are often of fundamental importance in proving existence or good behaviour of solutions to nonlinear equations, where the nonlinearity arises from advection terms. They should be accurately respected by numerical methods, but it is very difficult to achieve this in practice.

Section 3 deals with three-dimensional inviscid incompressible flow. The important extra effect included is vortex stretching. It is believed, though not proved, that this term in the equations can produce infinite vorticity in a finite time, and that this is the mechanism by which the scale of motion is reduced in three-dimensional turbulence so that viscous dissipation can balance the energy production, however small the viscosity. The extreme nonlinearity of this process and the consequent rapid reduction in scale means that it is grossly under-represented in conventional Eulerian numerical models. In situations where this effect is important, it may be impossible to obtain an accurate solution. While the dissipation can be parametrized by grid-scale viscosity, there is no way of ensuring that the large scale structure is correct. It is therefore important to identify special cases where this type of vortex stretching is not dominant, and resolvable long-lasting stable structures can exist. The theories in this area are discussed. It is important that any special conservation properties which ensure stable structures are accurately represented by numerical schemes.

Most atmospheric features of direct relevance to weather forecasting can be described quite accurately by simplified systems of equations, though it is unlikely that any one simplified system can describe all such features. The hydrostatic relation is very accurate on all scales that can be represented in global models either now or in the next 10 years. Numerical methods even for non-hydrostatic equations need to be able to represent a state of rest in hydrostatic balance exactly, which restricts the choice of approximation that can be used successfully over topography. In middle latitudes the geostrophic constraint is important outside the boundary layer. Numerical methods for the primitive equations need to be able to represent and maintain this balance in a straight flow. In more general flows, the ageostrophic or divergent flow can be calculated implicitly on making some scaling assumptions, for instance that certain time derivatives are small. It is necessary that if the same scaling assumptions are imposed on the numerical solution, the resulting implicit equations for the divergent or ageostrophic wind fields have solutions close to the exact solution. The evolution of the balanced flow is typically controlled by potential vorticity conservation in the absence of diabatic effects. The solution of the adiabatic balanced problem must consist at all times of a rearrangement of the initial potential vorticity. The numerical approximations to the relevant terms in the primitive equations should reflect this. Section 4 discusses the requirements imposed by various definitions of balanced motion. Section 5 reviews the resulting properties required from numerical methods for the primitive equations, and discusses possible compromises between the conflicting requirements.

2. ADVECTION AND REARRANGEMENT

2.1 The barotropic vorticity equation

First consider the simple problem of solving the barotropic vorticity equation:

$$\partial\omega/\partial t + \underline{u}.\nabla\omega = 0 \qquad (1)$$

$$\nabla.\underline{u} = 0. \qquad (2)$$

The equations are to be solved in a region Ω in R^2 with the normal component of $\underline{u}$ zero on the boundary $\delta\Omega$. It has been proved by Kato(1967) that, given an initially smooth distribution of the vorticity, the solution will remain smooth indefinitely. This is because the solution at any time has to be a rearrangement of the initial vorticity. The velocity field is determined from the vorticity field, the continuity equation, and the boundary conditions by solving a Poisson equation for the streamfunction. Inverting the Poisson operator acts as a smoother and hence it can be shown that the velocity field is smooth. The rearrangement of the vorticity is therefore also smooth, meaning that two initially separated fluid elements cannot come arbitrarily close together. The relation between this theorem and the phenomenological description of two-dimensional turbulence, which is governed by these equations, has been discussed by Vallis(1985). It is important to note that, though the problem is nonlinear and analytic solutions rarely available, the structure of the solution is very simple. This makes it easy to identify properties which should be satisfied by numerical methods.

A numerical solution of this problem should have a similar regularity theorem which would ensure complete nonlinear stability, and hence convergence to the analytic solution at all times as the resolution is increased. The existence of smooth solutions indicates that the ultimate convergence rate will be that predicted by a standard truncation error analysis. The proof of the theorem depends on three facts:

(i) The values of vorticity are bounded by their initial values,

(ii)The area of fluid associated with any value of the vorticity is conserved,

(iii)The stream function and hence the velocity field is determined from the vorticity by a Poisson equation whose solutions are controlled by a regularity result and a maximum principle.

(i) and (ii) together imply conservation of the integral of any function of the vorticity.

Property (iii) is easily enforced if a standard finite difference, finite element or spectral method is used to solve the Poisson equation. If, however, the inversion is done directly into velocity components which cannot be directly related to a discrete streamfunction, then the stability and convergence of this step cannot be guaranteed. This requires the use of the Arakawa C grid for finite difference schemes, or an appropriate choice of finite element approximation, see Thomasset(1981). Failure to observe this could lead to grid-scale instability requiring filtering.

Exact enforcement of both (i) and (ii) is impossible unless a special Lagrangian method is used in which the vorticity is assigned a discrete set of values initially which are then simply rearranged during the time integration. The closest approach to this is the vortex 'blob' method of Chorin (1980). THere are still difficulties with (iii) because the implied vorticity is not smooth and the method has to be modified to allow regularity results to be proved. Such methods are difficult to extend to more general problems and it is therefore necessary to compromise. Property (i) can be satisfied by upstream differencing, but this violates property (ii) by increasing the area assigned to the intermediate values of ω and reducing that assigned to the extreme values. More accurate versions of upstream differencing, e.g. Smolarkiewicz (1984), or semi-Lagrangian methods, Robert (1982), Bates and MacDonald (1982) may enforce (i) with acceptably small errors in (ii). The Arakawa schemes conserve the first and second moments of ω as an attempt to conserve all the moments. This enforces neither (i) or (ii) exactly but the conservation of two moments should prevent excessive errors.

2.2 Buoyancy driven flow

The second example of a problem whose solutions are controlled by a rearrangement property is that of two-dimensional incompressible flow in a vertical cross-section:

$$\partial u/\partial t + \underline{u}.\nabla u + \partial p/\partial x = 0 \tag{3}$$

$$\partial w/\partial t + \underline{u}.\nabla w + \partial p/\partial z - g\theta/\theta_o = 0 \tag{4}$$

$$\partial \theta/\partial t + \underline{u}.\nabla \theta = 0 \tag{5}$$

$$\partial u/\partial x + \partial w/\partial z = 0. \tag{6}$$

The equations are to be solved in a region Ω in R^2 with coordinates (x,z) and velocity $\underline{u} = (u,w)$. θ is the buoyancy. The boundary condition is that there is no normal flow. A scalar vorticity can still be defined which governs the evolution but is not conserved. It obeys the equation

$$\partial \omega/\partial t + \underline{u}.\nabla \omega - (g/\theta_o)\partial \theta/\partial x = 0 \tag{7}$$

These equations conserve energy and buoyancy. The solutions are controlled by the requirement that θ is an area-preserving rearrangement of its initial values. Given smooth initial data for $\underline{u}$ and θ, $\partial\theta/\partial x$ is bounded and so ω can only change at a bounded rate. While ω remains bounded, the velocity field must stay smooth and, in particular, fluid parcels initially a finite distance apart cannot come arbitrarily close together. This means that $\partial\theta/\partial x$ will remain bounded. In principle this should allow a proof that the solutions remain smooth for arbitrary finite time given smooth initial data; this argument has not yet, however, been turned into a rigorous proof.

It is important that numerical approximations to this system contain a good treatment of the unforced problem obtained by setting $g=0$. This is because the property that initially separated fluid parcels remain separated is needed to restrict the value of $\partial\theta/\partial x$ and hence the growth in vorticity. In addition the rearrangement property of θ must be respected. This requires either accurate upstream or semi-Lagrangian differencing, or a scheme that conserves the integrals of θ and θ^2.

It is not likely that energy conservation alone is a guarantee of good behaviour as can be illustrated by the following example. Given an initial atate of rest in hydrostatic balance, the energy E can be written

$$E = E_0 - \int g\theta z/\theta_0 \, d\Omega, \quad (8)$$

where E_0 is a reference value. The minimum energy state possible by rearrangement of the θ values according to (5) is with θ a function of z only and with θ increasing with z. If a small perturbation energy δE is added, and the equations integrated, an energy conserving scheme will maintain a total energy of $E+\delta E$. However, if the rearrangement property of θ is not exactly enforced, the solution may be able to extract energy from the supposedly minimised basic state by increasing the mean stratification and converting the unphysical extra energy into kinetic energy.

3. THREE DIMENSIONAL INCOMPRESSIBLE FLOW

The equations can be written

$$\partial\underline{\omega}/\partial t + \underline{u}.\nabla\underline{\omega} + \underline{\omega}.\nabla\underline{u} = 0 \quad (9)$$

$$\nabla.\underline{u} = 0 \quad (10)$$

in a region Ω in R^3 with no normal flow across the boundary $\delta\Omega$. Thses allow vorticity to be generated by stretching as well as advected. The kinetic energy is conserved. If the advection of the passive scalar θ is included in the model, so that

$$\partial\theta/\partial t + \underline{u}.\nabla\theta = 0, \quad (11)$$

then the potential vorticity $\underline{\omega}.\nabla\theta$ is conserved following the motion.

These constraints on energy and potential vorticity do not guarantee good behaviour of the system. The work of Chorin (1982), Ballard (1985), and Vallis (1985), amongst others, all supports the idea that the solutions can become singular through production of an infinite amount of vorticity in a finite time. It has not been possible to prove this rigorously. The mechanism is believed to be the existence of configurations of vorticity in which $\underline{\omega}.\nabla\underline{u}$ is proportional to ω^2. Equation (9) then behaves like the equation

$$\partial\omega/\partial t = \omega^2 \qquad (12)$$

whose solutions become infinite at time t=1. As discussed by Vallis, this singularity corresponds to the scale collapse in three dimensional turbulence which allows viscous dissipation to balance the energy production, however small the viscosity. It was shown by Ballard that, in order to maintain energy conservation, the increase in vorticity primarily occurs by the production of paired 'filaments' of opposite signed vorticity, thus allowing the induced velocity to cancel. The potential vorticity conservation does not constrain the flow at all because $\nabla\theta$ along a filament tends to zero at the same rate as the vorticity tends to infinity. The conservation is then destroyed by viscosity.

This process involves a catastrophic scale collapse and cannot be effectively simulated by Eulerian numerical methods because a sufficient range of scales cannot be simulated accurately. The vortex stretching term can be computed accurately while the resolution is sufficient, but its effect on reducing the scale is grossly underestimated. The solutions obtained are more like those for three-dimensional vorticity advection, as illustrated by Ballard(1986). This difficulty is also known in engineering calculations, where turbulent wakes in a flow are often shorter than would be expected from computer simulations which underestimate the vortex stretching and hence the dissipation. The simulations using vortex methods by Chorin and Ballard are not entirely reliable either, though they indicate the probable mechanism of scale collapse.

The implications of this work for numerical methods is rather negative, because an accurate simulation of this process is impossible. All that can be done is to insert the dissipation at the grid scale, allowing a stable but inaccurate solution. A particular danger is that vorticity will be advected rather than destroyed, leading to the presence of apparently stable but spurious structure in the solution. In situations where coherent structure is observed in the atmosphere, vortex stretching must be inhibited and accurate prediction may be possible. It is therefore important to identify such situations, and to ensure that the reasons why vortex stretching are inhibited are faithfully reflected in the numerical approximation.

One such situation is the Beltrami flow. This has the velocity and the vorticity parallel everywhere. In this case $\underline{u} \times \underline{\omega}$ is zero, and $\underline{u}.\nabla\underline{u}$ can be written as $\nabla(\frac{1}{2}\underline{u}^2)$. Hence $\partial\underline{\omega}/\partial t$ is zero and also $\partial\underline{u}/\partial t$. It is not known how stable such configurations are. They are characterised by high helicity $\underline{u}.\underline{\omega}$. Lilly (1986) suggests that high helicity inhibits vortex stretching sufficiently to produce enhanced predictability in practice, while Kraichnan (1973) argues that helicity has little effect on the structure of turbulence.

If helicity is important in real flows it should be reflected in the parametrizations of turbulence in three-dimensional models. Since the instability due to vortex stretching is likely to be under-represented, it is likely to be possible to model long-lived helical structures. The spurious structures which may be retained due to lack of vortex stretching should be removed by the turbulence model. The helicity could be used as an input to the turbulence model, reducing its effect when the helicity is large.

Haynes and McIntyre(1987) have pointed out various conservation properties that result from the fact that the vorticity is the curl of another vector. This implies that care is needed in the choice of numerical approximations if the vorticity is used as a variable. As yet the detailed requirements have not been worked out. If the equations are written as evolution equations for the velocity rather than the vorticity, then there is no difficulty since their results do not imply any restriction on the velocity field.

4. BALANCED EQUATIONS

4.1 The need to consider balanced models

The most important motions for weather forecasting can be regarded as balanced in some sense, since the observed extra-tropical flow outside the boundary layer is close to geostrophic and high frequency solutions anywhere are not relevant. Recent work on balance, Cullen et al. (1987), illustrates that a wide variety of flows can be described by a balanced model, including those where convection is important. It is unlikely, however, that a single definition of balance can cover all flows of interest. When designing numerical methods for a primitive equation model to be used in operational forecasting, it is necessary to ensure that the numerical model contains balanced subsystems which are accurate models of the equivalent continuous subsystems. It is less important to integrate model gravity waves accurately since only a small part of the atmospheric gravity wave spectrum can be resolved and the mechanisms which force the waves cannot be well represented.

There is no general agreement as to what definition of balance is most useful. If the concept is restricted to the elimination of high frequencies, then it can only be defined for smooth vertical structures, Browning and Kreiss (1986). Normal mode initialisation is usually only applied to smooth vertical structures and is not used to eliminate internal gravity waves with large vertical wavenumber. The semi-geostrophic definition, Cullen and Norbury (1986), allows complete removal of gravity waves, as do the nonlinear balanced equations reviewed by Gent and McWilliams (1983). The implications of a variety of definitions for numerical methods is therefore described. In this section we consider integration schemes for pure balanced models, and then in section 5 discuss the implications for primitive equation models,

4.2 Quasi-geostrophic models

First consider the barotropic vorticity equation with variable Coriolis parameter:

$$\partial\omega/\partial t + \underline{u}.\nabla\omega + \beta v = 0 \quad (12)$$

$$\nabla.\underline{u} = 0. \quad (13)$$

In section 2 we discussed the implication that vorticity is simply rearranged by the advection term. A given smooth vorticity field induces a smooth and bounded velocity, and therefore βv is also smooth and bounded. Though the vorticity will no longer be bounded by its initial values, its rate of growth is restricted, This restriction will be well approximated if the numerical approximation to the advection term satisfies the rearrangement property discussed in section 2.1.

Now consider three-dimensional quasi-geostrophic flow. The equations can be written, following the notation of Hoskins (1975) and Gent and McWilliams (1983):

$$\partial u_g/\partial t + \mathbf{u_g}.\nabla u_g + \partial p/\partial x - f_o v - \beta y v_g = 0 \quad (14)$$

$$\partial v_g/\partial t + \mathbf{u_g}.\nabla v_g + \partial p/\partial y + f_o u + \beta y u_g = 0 \quad (15)$$

$$\partial\theta/\partial t + \mathbf{u_g}.\nabla\theta + w\partial\theta^o/\partial z = 0 \quad (16)$$

$$f u_g + \partial p/\partial y = 0 \quad (17)$$

$$f v_g - \partial p/\partial x = 0 \quad (18)$$

$$g\theta/\theta_o - \partial p/\partial z = 0 \quad (19)$$

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \quad (20)$$

The equations are to be solved in a closed region Ω in $\mathbf{R}^3$ with coordinates (x,y,z). The boundary conditions will be discussed later. θ is the buoyancy and $\theta^o(z)$ a reference profile. The Coriolis parameter is $(f_o + \beta y)$ where $\beta y \ll f_o$. In the oceanic context z is a height and p a pressure, for a Boussinesq atmosphere z is a function of pressure and p is a geopotential. These equations give a potential vorticity equation:

$$q \equiv \omega_g + f\partial\theta/\partial z/(\partial\theta^o/\partial z) - \beta y \quad (21)$$

$$\omega_g \equiv \partial v_g/\partial x - \partial u_g/\partial y \quad (22)$$

$$\partial q/\partial t + \mathbf{u_g}.\nabla q = 0. \quad (23)$$

q can be written in terms of p as:

$$q = \partial/\partial x(f^{-1}\partial p/\partial x) + \partial/\partial y(f^{-1}\partial p/\partial y) + (f\theta_o/g)\partial^2 p/\partial z^2/(\partial\theta^o/\partial z) - \beta y. \quad (24)$$

Bennett and Kloeden(1981) proved existence and uniqueness for two versions of these equations. The proof depends on recognising that potential vorticity q is simply rearranged by the geostrophic wind, and the geostrophic wind is derived from q by solving the Poisson equation (24) for p. θ is then determined from (19) and the total velocity components (u,v,w) need never be calculated though they can be diagnosed from (15), (14), and (16) respectively. Boundary conditions are required to allow (24) to be solved. A sufficient set are to specify either the normal derivative $\partial p/\partial n$ or p on the boundaries. As discussed by Bennett and Kloeden, care is needed in specifying a consistent set, but this aspect is not directly relevant to numerical methods for global primitive equation models.

The key property for numerical methods is again the rearrangement property for q, and the good behaviour of $\underline{u}_g$ derived from it by solving (24). The natural numerical method would be to use q as a primary variable. Schemes for advecting it should be selected for their rearrangement properties as discussed in section 2.1. In a finite difference scheme, it is natural to hold q and p at the same gridpoints. The geostrophic wind components are then held on the Arakawa 'D' grid, Haltiner and Williams (1980). The potential temperature is staggered in the vertical from the winds as suggested by Charney and Phillips (1953). q and p are held either at boundary points where p is specified or half a gridlength away when $\partial p/\partial n$ is specified. If finite element methods are used it is natural to use the same basis functions for q and p, If the advection of q is written in the Jacobian form, conservation of the integrals of q and q^2 then follows directly because of the properties of the Galerkin method.

4.3 Semi-geostrophic models

The equations are:

$$\partial u_g/\partial t + \underline{u}.\nabla u_g + \partial p/\partial x - fv = 0 \tag{25}$$

$$\partial v_g/\partial t + \underline{u}.\nabla v_g + \partial p/\partial y + fu = 0 \tag{26}$$

$$\partial \theta/\partial t + \underline{u}.\nabla \theta = 0 \tag{27}$$

$$\underline{u} = (u,v,w), \tag{28}$$

together with (17) to (20). The equations are to be solved in a closed region Ω in $\mathbb{R}^3$ with no normal flow through the boundary. Existence and uniqueness of solutions to a finite-dimensional approximation to this problem were proved by Cullen and Purser (1984).

The key step in the theory for these equations is the method of constructing **u**. This is done for constant f by seeking a volume-preserving rearrangement of θ, $(u_g - fy)$, and $(v_g + fx)$ which is consistent with the geostrophic and hydrostatic relations. Cullen and Purser prove that there is a unique such rearrangement which has non-negative potential vorticity.

In a finite difference model, Cullen(1987) showed that the appropriate arrangement of variables is the same as the quasi-geostrophic model for u_g, v_g and θ. The total wind components are held on the 'C' grid with w staggered in the vertical from u and v. u is then at the same points as v_g, v at the same points as u_g and θ at the same points as w. The schemes used to advect u_g, v_g and θ should satisfy the rearrangement property. If a Galerkin finite element method is used the correct choice of approximating functions is difficult. The horizontal part of the total velocity **u** derived from the implicit calculation has both rotational and divergent components, so that if the streamfunction and velocity potential are used as variables, they have to use different node-points. If the velocity components are used, they also have to be approximated using different basis functions. Either choice presents difficulties on an irregular mesh.

The finite difference implementation is described in Cullen(1987). Since the true solution may be discontinuous, it is necessary to iterate towards the required value of **u**. Because the desired rearrangement is unique, this iteration is guaranteed to converge to it provided that the numerical method satisfies the rearrangement property and the potential vorticity is not allowed to become negative. If negative potential vorticity is generated, an adjustment which satisfies the rearrangement property must be carried out.

4.4 Nonlinear balance equation models

There are a number of versions of these. We select the BE model of Gent and McWilliams(1983) for illustration. The equations are (17) to (20) and (27) to (28) together with:

$$\nabla.(f\nabla\psi) + 2J(\psi_x, \psi_y) = \nabla^2\varphi \qquad (29)$$

$$D/Dt(f+\nabla^2\psi) + (f+\nabla^2\psi)\nabla^2\chi + \nabla w.\nabla\psi_z = 0. \qquad (30)$$

The choice of boundary conditions is not straightforward and is discussed by Gent and McWilliams. ψ and χ are the streamfuction and velocity potential for the horizontal velocity.

The structure of the equations differs slightly from the semi-geostrophic in that only the velocity potential has to be predicted implicitly from an omega equation.The streamfuction can be stepped forward explicitly. An ellipticity condition is required to allow a solution to be found. There is as yet no existence proof for these equations. However, both θ and $\nabla^2\psi$ are advected, and the resulting rearrangement property would be an important element of any proof.

The choice of staggering for a finite difference model or of basis functions for a finite element model is easier than for the semi-geostrophic case. The streamfunction, geopotential, and velocity potential should be defined at the same points. The Jacobian form can be used for the advection by the rotational wind to ensure conservation of first and second moments of $\nabla^2\psi$. The rearrangement property for θ should also be preserved.

4.5 Nonlinear normal-mode initialisation and the bounded derivative equations

Many primitive equation models are initialised by the normal mode method. This is usually only applied to those vertical modes whose associated gravity wave frequencies are large compared with those associated with advection. This procedure can only be used to define a balanced model if the integration scheme uses the normal modes, as does that of Daley(1980).

Temperton(1986) has shown that similar behaviour can be obtained from a scheme which does not use the normal modes directly.This approach which can be used as the basis of an integration scheme using finite difference or finite element methods. Another such scheme is the bounded derivative method of Browning and Kreiss (1986). They prove that the requirement that all fields evolve on the advective timescale or slower can be used to give a definition of balance for smooth vertical structures. The exclusion of gravity waves with large vertical wavenumber cannot be justified by this method. The resulting reduced systems of equations describing balance have similarities in structure to the nonlinear balance equation, so that the implications for numerical methods are like those discussed in section 4.4.

5. BALANCED SOLUTIONS OF PRIMITIVE EQUATION MODELS

The primitive hydrostatic equations in the notation of section 4 are (19) to (20),(27) to (28) and:

$$Du/Dt + \partial p/\partial x - fv = 0 \tag{31}$$

$$Dv/Dt + \partial p/\partial y + fu = 0. \tag{32}$$

The non-hydrostatic equations replace (19) by

$$Dw/Dt + \partial p/\partial z - g\theta/\theta_0 = 0. \tag{33}$$

The Boussinesq approximation can be withdrawn by replacing (20) by

$$\partial u/\partial x + \partial v/\partial y + \partial(r(z)w)/\partial z = 0, \tag{34}$$

where r is a known function of z.The lower boundary condition is:

$$w = \partial z_*/\partial t + \underline{u}_*.\nabla z_* \text{ at } z=z_B \tag{35}$$

$$\partial z_*/\partial t + \underline{u}_*.\nabla z_* = -\int_0^H(\partial u/\partial x + \partial v/\partial y)\ dz. \tag{36}$$

w is zero at the upper boundary corresponding to zero pressure.

There is a close analogy between solving the primitive equations by semi-implicit methods and solving balanced equations. The standard semi-implicit method uses the pressure gradient terms and the vertical advection of θ at the new time level. The procedure can be simply illustrated:

$$u^{t+\Delta t} + \Delta t(\partial p/\partial x)^{t+\Delta t} = A^t \tag{37}$$

$$v^{t+\Delta t} + \Delta t(\partial p/\partial y)^{t+\Delta t} = B^t \tag{38}$$

$$w^{t+\Delta t} + \Delta t(\partial p/\partial z - g\theta/\theta_0)^{t+\Delta t} = C^t \tag{39}$$

$$\theta^{t+\Delta t} + \Delta t w^{t+\Delta t}\partial\theta/\partial z = D^t. \tag{40}$$

Equations (34) to (36) are also applied at time t+Δt. Substituting (40) into (39) to eliminate θ gives

$$\partial p/\partial z^{t+\Delta t} + \Delta t w^{t+\Delta t}(1 - (g/\theta_0)\partial\theta/\partial z^t) = E^t. \tag{41}$$

Eliminating p between (37) and (41) and between (38) and (41) gives equations for velocities at time t+Δt of the form

$$\partial u/\partial z + \Delta t^2(F\partial w/\partial x + Gw) = H \tag{42}$$

$$\partial v/\partial z + \Delta t^2(I\partial w/\partial y + Jw) = K. \tag{43}$$

Use of (34) to eliminate u and v gives

$$-\partial^2(r(z)w)/\partial z^2 + \Delta t^2[\partial/\partial x(F\partial w/\partial x + Gw) + \partial/\partial y(I\partial w/\partial y + Jw)] = L. \tag{44}$$

This is an elliptic equation for w of a similar form to an omega equation. When used in primitive equation models it is usual to replace $\partial\theta/\partial z$ in equation (41) by a reference value independent of x and y. This removes the terms denoted G and J in (41) and allows the equations to be decomposed into two-dimensional equations when the vertical derivative is approximated by finite differences. The nonlinear balance equation of section 4.4 is obtained by removing the time derivative from equation (33) and the divergence of equations (31) and (32). This removes the first term in (44) and alters the form of F, G, I and J; also reducing the equation to a two-dimensional equation:

$$\partial/\partial x(F^b\partial w/\partial x + G^b w) + \partial/\partial y(I^b\partial w/\partial y + J^b w) = L. \tag{45}$$

Equation (45) can be considered as an approximation to the omega equation derived from the nonlinear balance equation in which the most rapidly varying terms are approximated at time $t+\Delta t$ and the rest at time t.

If the balanced part of the flow is to be well treated by the primitive equation model, it is desirable that at least all the terms on the left hand side of (45) should be treated implicitly, giving a variable coefficient elliptic equation. The approximation should be chosen so that the regularity properties implied by (45) are preserved. This means that the streamfunction, velocity potential and geopotential should all be held at the same points or approximated by the same finite element basis. There is no ideal staggering if velocity components are used as variables. The Arakawa E grid appears to satisfy the requirement. However, if an approximation to (45) is generated from this staggering, the Laplacian operator in it is not approximated using the nearest gridpoints. This allows checkerboard instabilities to develop.

A further complication is the ellipticity condition. If this is violated, then the balanced omega equation cannot be solved. The primitive equation form, equation (44), can be solved. The actual flow will then be unbalanced and may be difficult to treat properly because of limited resolution. For instance, in the semi-geostrophic case, the ellipticity condition can be stated as requiring non-negative potential vorticity. Regions violating this condition can be generated by latent heat release. The real flow then contains overturning motions including a lot of small scale activity. It is arguable that such situations should be excluded by a parametrization scheme even when the primitive equations are integrated. Cullen et al. (1987) show an example of a forecast generating ever increasing vertical velocities as resolution is increased when the potential vorticity is negative.

The elliptic equation (45) generates vertical motions which are roughly inversely proportional to th static stability. Considerable increases in accuracy might be possible if the moist static stability were used in (45) when appropriate. This would achieve a closer coupling between moist effects and the rest of the dynamics, particularly in rapid cyclogenesis. If this is done all saturated regions of negative moist potential vorticity must be removed by a parametrization.

6. CONCLUSIONS

The study of the properties of the partial differential equations presented here gives conditions for nonlinear stability of integration schemes but only limited information about accuracy. Nonlinear stability requires:

- The rearrangement property
- Correct choice of approximating functions or grid staggering for different variables.

The rearrangement property cannot be satisfied exactly by a conventional scheme, but can be approximated by

- Conserving moments
- Accurate upstream differencing
- Reduction of truncation error generally

Research is needed into which schemes are best in this respect. This can be done by calculating the evolution of a number of the moments in standard advection tests.

The best positioning of variables appears to be to have the vorticity, divergence and geopotential at the same points. This is normal in spectral and finite element models. There is no completely satisfactory arrangement if velocity components are approximated, perhaps explaining why there is disagreement about which scheme is best. Potential temperature should be staggered in the vertical from the geopotential, as has been suggested many times but rarely implemented. An alternative strategy is to filter all grid-scale information. However, it has never been proved that this is sufficient to ensure nonlinear stability.

Accurate solution of the balanced part of the flow requires solution of a variable coefficient elliptic equation, conventional semi-implicit methods solve a constant coefficient equation. There may be advantages in going to a more fully implicit approach, requiring development of three-dimensional variable coefficient elliptic solvers. Such methods should use the moist rather than the dry static stability in saturated regions of ascent.

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