Application of the normal modes of a three-dimensional primitive equations model to data analysis and forecasting

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ABSTRACT

In contrast to the empirical orthogonal functions (eof's) in which the physics of the atmosphere is reflected statistically, normal mode functions (NMFs) are the eigensolutions to free oscillations of a linearized atmospheric model. Both EOFs and NMFs are useful in the analysis of atmospheric data and are applicable to formulate a prediction model. This note discusses the formulation of the simplest NMFs for a global primitive equations model and their application to analysis and prediction of the three-dimensional global atmosphere.

1. Introduction

The expansion of meteorological data in terms of empirically defined orthogonal functions is efficient, because empirical orthogonal functions (EOFs) minimize the root-mean-square difference between the data and the functional representation (e.g., Lorenz, 1956; Obukhov, 1960; Holmström, 1963). The physics of the atmosphere is reflected statistically in the characteristics of EOFs. With respect to the vertical structure of transient motions, there have been several attempts to "interpret" the characteristics of EOFs based on atmospheric equations (Holmström, 1964; Gavrilin, 1965; Simons, 1968; Wiin-Nielsen, 1971; McFarlane, 1971; Bodin, 1974; Baer, 1974 and Kasahara, 1976).

In the representation of meteorological data, we are interested in another approach--application of normal mode functions--which is closely related to the EOF approach. Expansion functions are the eigensolutions to free oscillations of a linearized primitive equations model, referred to as normal mode functions (NMFs). For details, see Flattery (1970), Dickinson and Williamson (1972) and Kasahara (1976).

So far, the expansion of data in terms of NMFs is principally applied to two-dimensional hemispheric or global data. The expansion of three-dimensional data in terms of three-dimensional NMFs was performed by Penenko (1974) and Williamson and Dickinson (1976). Penenko constructed NMFs by the finite-difference method based on a three-dimensional linearized primitive equations model in which basic zonal flow, temperature and pressure were functions of height, latitude and longitude. Williamson and Dickinson (1976) also constructed NMFs by the finite-difference method based on a three-dimensional linearized primitive equations model with a basic state at rest and the basic temperature and pressure distributions depending on height only.

In this note, we present the formulation of three-dimensional NMFs based on the continuous form of linearized atmospheric equations rather than on the discrete form. In explaining the basic idea of the NMF approach, this presentation is considerably simpler than the finite-difference representation.

The construction of three-dimensional NMFs is described in Sections 2 through 5. The vertical structures of geopotential, density, vertical motion and temperature in an isothermal basic atmosphere are discussed in detail in Section 5. The expansion of three-dimensional data in terms of NMFs is presented in Section 6. The application of the NMF expansion to formulate a completely spectral baroclinic model is briefly commented on in Section 7. A future plan related to this area of research is presented in Section 8.

2. Basic equations

We consider motions of small amplitude superimposed on the basic state at rest, with temperature $\overline{T}$ as a function of height z only. Linearized equations of horizontal motion, hydrostatic equilibrium, mass continuity and thermodynamics in spherical coordinates with longitude λ, latitude ϕ and height z above mean sea level and time t are given as

$$\frac{\partial u'}{\partial t} - 2\Omega\sin\phi v' = -\frac{1}{\overline{\rho}a\cos\phi}\frac{\partial p'}{\partial\lambda}, \tag{2.1}$$

$$\frac{\partial v'}{\partial t} + 2\Omega\sin\phi u' = -\frac{1}{\overline{\rho}a}\frac{\partial p'}{\partial\phi}, \tag{2.2}$$

$$\frac{\partial p'}{\partial z} = -\rho' g, \tag{2.3}$$

$$\frac{\partial \rho'}{\partial t} + w'\frac{d\overline{\rho}}{dz} + \overline{\rho}\nabla\cdot\mathbb{V}' + \overline{\rho}\frac{\partial w'}{\partial z} = 0, \tag{2.4}$$

$$\frac{\partial p'}{\partial t} + w'\frac{d\overline{p}}{dz} = \gamma g\overline{H}\left(\frac{\partial \rho'}{\partial t} + w'\frac{d\overline{\rho}}{dz}\right), \tag{2.5}$$

where

$$\nabla\cdot\mathbb{V}' = \frac{1}{a\cos\phi}\left[\frac{\partial u'}{\partial\lambda} + \frac{\partial}{\partial\phi}(v'\cos\phi)\right] \tag{2.6}$$

denotes the horizontal divergence and

$$\overline{H} = R\overline{T}/g$$

denotes the scale height of the basic atmosphere.

In Eqs. (2.1)-(2.5), u, v, p, ρ and w are, respectively, the eastward and northward velocity components, pressure, density and vertical velocity. The prime denotes a perturbation variable and the overbar refers to a basic-state quantity. Also, Ω, a and g denote the earth's angular velocity, radius and gravity, respectively. Quantity γ represents C_p/C_v, the ratio of two specific heat coefficients at constant pressure and constant volume, and $R(= C_p - C_v)$ is the specific gas constant.

3. Separation of variables

As suggested by Taylor (1936), it is convenient to introduce the separation constant D_n to satisfy

$$\frac{\partial p'}{\partial t} + \bar{\rho} g D_n \nabla \cdot \mathbb{V}' = 0 \ , \tag{3.1}$$

where D_n has the dimension of height and is later identified as the equivalent height. In regard to the historical development related to the concept of equivalent height, the reader may refer to, for example, Wilkes (1949), Siebert (1961) and Chapman and Lindzen (1970).

Using (3.1), we rewrite Eqs. (2.4) and (2.5) as

$$\frac{\partial \rho'}{\partial t} - \frac{1}{gD_n}\frac{\partial p'}{\partial t} = -\bar{\rho}\frac{\partial w'}{\partial z} + \frac{1}{\bar{H}}\left(1 + \frac{d\bar{H}}{dz}\right) w' \ , \tag{3.2}$$

and

$$\frac{\partial p'}{\partial t} - \gamma g \bar{H}\frac{\partial \rho'}{\partial t} = \bar{\rho} g w' - \gamma g\left(1 + \frac{d\bar{H}}{dz}\right) w' \ , \tag{3.3}$$

in which the following relationship is used:

$$\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dz} = -\frac{1}{\bar{H}}\left(1 + \frac{d\bar{H}}{dz}\right) . \qquad (3.4)$$

Eqs. (3.2) and (3.3) consist of two equations for $\partial p'/\partial t$ and $\partial \rho'/\partial t$ which can be solved as functions of w'. However, ρ' and p' must satisfy the hydrostatic constraint (2.3) all the time and this leads to a diagnostic equation for w', known as Richardson's equation. Once p' and ρ' are determined, the temperature perturbation can be calculated from the equation of state:

$$T'/\bar{T} = p'/\bar{p} - \rho'/\bar{\rho} . \qquad (3.5)$$

We are now ready to specify the vertical dependence of the perturbation variables:

$$\left.\begin{aligned} u' &= \hat{u}\Phi_n(z) , & w' &= (\partial\hat{h}/\partial t)\, W_n(z) , \\ v' &= \hat{v}\Phi_n(z) , & \rho'/\bar{\rho} &= (\hat{h}/D_n)\, R_n(z) , \\ p' &= g\bar{\rho}\hat{h}\Phi_n(z) , & T'/\bar{T} &= (\hat{h}/D_n)\, T_n(z) . \end{aligned}\right\} \qquad (3.6)$$

Here, Φ_n, W_n, R_n and T_n are dimensionless functions of height only. On the other hand, $\hat{u}$, $\hat{v}$ and $\hat{h}$ depend on latitude ϕ, longitude λ and time t and satisfy

$$\left.\begin{aligned} \frac{\partial\hat{u}}{\partial t} - 2\Omega\sin\phi\hat{v} &= -\frac{g}{a\cos\phi}\frac{\partial\hat{h}}{\partial\lambda} , \\ \frac{\partial\hat{v}}{\partial t} + 2\Omega\sin\phi\hat{u} &= -\frac{g}{a}\frac{\partial\hat{h}}{\partial\phi} , \\ \frac{\partial\hat{h}}{\partial t} + D_n\nabla\cdot\mathbb{V}' &= 0 , \end{aligned}\right\} \qquad (3.7)$$

which are known as linearized "shallow-water" equations for the mean fluid depth of D_n.

4. Vertical structure of variables

Substitution of (3.6) into (3.2) and (3.3) and solution of $R_n(z)$ and $\Phi_n(z)$ from the resulting two equations in terms of $W_n(z)$ yield

$$\left(1 - \gamma \frac{\bar{H}}{D_n}\right) \Phi_n = W_n - \gamma\bar{H} \frac{dW_n}{dz} , \tag{4.1}$$

$$\left(1 - \gamma \frac{\bar{H}}{D_n}\right) R_n = \left[1 - \left(\gamma - \frac{D_n}{\bar{H}}\right) \left(1 + \frac{d\bar{H}}{dz}\right)\right] W_n - D_n \frac{dW_n}{dz} . \tag{4.2}$$

Similarly, the hydrostatic equation (2.3) is expressed by

$$\frac{d\Phi_n}{dz} - \frac{1}{\bar{H}} \left(1 + \frac{d\bar{H}}{dz}\right) \Phi_n = - \frac{R_n}{D_n} . \tag{4.3}$$

Now, Φ_n and R_n given by (4.1) and (4.2) must satisfy (4.3) and this leads to a diagnostic equation for W_n. That is

$$\frac{d^2W_n}{dz^2} + \left[\frac{\gamma}{(D_n - \gamma\bar{H})} \frac{d\bar{H}}{dz} - \frac{1}{\bar{H}}\right] \frac{dW_n}{dz}$$

$$+ \left[\frac{\kappa}{D_n\bar{H}} - \frac{\gamma}{D_n(D_n - \gamma\bar{H})} \frac{d\bar{H}}{dz}\right] W_n = 0 , \tag{4.4}$$

where $\kappa = (\gamma - 1)/\gamma$.

With the boundary conditions that W_n vanish at $z = 0$, the mean sea level, and $z = z_T$, the top boundary, i.e.,

$$W_n(o) = W_n(z_T) = 0 , \tag{4.5}$$

Eq. (4.4) forms an eigenvalue problem to determine the eigenvalue D_n.

The eigenfunction W_n gives the vertical structure of vertical motion. Once W_n is obtained, the vertical structures of Φ_n and R_n are determined from (4.1) and (4.2). The vertical structure of T_n is determined from

$$T_n = (D_n/\overline{H})\ \Phi_n - R_n\ , \tag{4.6}$$

which is readily obtained after substitution of (3.6) into (3.5).

5. Case of isothermal atmosphere

It is instructive to obtain the explicit solutions of W_n, Φ_n, R_n and T_n for an isothermal atmosphere $\overline{T}$ = constant. Since $\overline{H}$ = constant, Eq. (4.4) is reduced to

$$\frac{d^2W_n}{dz^2} - \frac{1}{\overline{H}}\frac{dW_n}{dz} + \frac{\kappa}{D_n\overline{H}} W_n = 0\ . \tag{5.1}$$

5.1 External mode, n = 0

As seen from (4.1) and (4.2), the case of D_n being equal to $\gamma\overline{H}$ is singular and must be treated separately. We designate it by n = 0 so that

$$D_o = \gamma\overline{H} \tag{5.2}$$

is the value of equivalent depth for this case. It turns out that this case leads to the following vertical structure:

$$\begin{aligned} W_o(z) &= 0\ , \\ \Phi_o(z) &= \exp(\kappa z/\overline{H})\ , \\ R_o(z) &= \Phi_o(z)\ , \\ T_o(z) &= (\gamma - 1)\ \Phi_o(z)\ . \end{aligned} \tag{5.3}$$

Because this mode is characterized by having no vertical motion and the geopotential profile by having no node, we refer to it as an external mode.

5.2 Internal modes, $n \geqslant 1$

The solution $W_n(z)$ of (5.1), satisfying the boundary conditions (4.5), is given by

$$W_n(z) = \exp\left(\frac{1}{2} z/\overline{H}\right) \sin(mz) \quad , \tag{5.4}$$

where

$$m = \left[\frac{\kappa}{D_n \overline{H}} - \frac{1}{4\overline{H}^2} \right]^{\frac{1}{2}} \tag{5.5}$$

which is real. The eigenvalue D_n of Eq. (5.1) corresponding to solution (5.4) becomes

$$D_n = \frac{4\kappa\overline{H}z_T^2}{z_T^2 + 4n^2\pi^2\overline{H}^2} \quad . \tag{5.6}$$

The vertical structures of R_n, Φ_n and T_n are calculated from

$$\begin{aligned} R_n(z) &= -\left[\left(1 - \gamma + \frac{D_n}{\overline{H}}\right) W_n - D_n \frac{dW_n}{dz}\right] \Big/ \left(\frac{D_o}{D_n} - 1\right) , \\ \Phi_n(z) &= \left(W_n - D_o \frac{dW_n}{dz}\right) \Big/ \left(\frac{D_o}{D_n} - 1\right) , \\ T_n(z) &= \frac{D_n}{\overline{H}} \Phi_n(z) - R_n(z) \quad . \end{aligned} \tag{5.7}$$

As the mean temperature $\overline{T}$, we assume that $\overline{T} = 243.90°K$ which gives the equivalent height corresponding to the external mode as $D_o = 10$ km, which is now representative (e.g., Lindzen and Blake, 1972).

Table 1 shows the values of parameters pertinent to this calculation. Column 1 gives the vertical mode index. Column 2 shows the vertical scale in kilometers measured by the height distance between two consecutive nodes in $W_n(z)$. Column 3 gives the equivalent height in meters for the present isothermal atmosphere. Columns 4 and 5 show the equivalent heights corresponding to a realistic temperature distribution with the tops at 18 and 36 km, respectively, obtained by the finite-difference method (Kasahara, 1976). The values of D_n for internal modes fall between D_n^* and D_n^{**}. Column 6 gives the gravity wave speed for D_n. Column 7 shows the corresponding values of Lamb's parameter ε.

Figs. 1 and 2 illustrate the vertical structures of $R_n(z)$, $\Phi_n(z)$, $T_n(z)$ and $W_n(z)$ for the present isothermal atmosphere. The top of the atmosphere is assumed to be 18 km.

5.3 Normal mode functions (NMFs)

The elementary solutions of (2.1)-(2.5) are now expressed by

$$\left.\begin{aligned} u' &= (gD_n)^{\frac{1}{2}}\, U_n(\phi)\ \Phi_n(z) \,, \\ v' &= -i(gD_n)^{\frac{1}{2}}\, V_n(\phi)\ \Phi_n(z) \,, \\ p' &= g\bar{\rho}D_n Z_n(\phi)\ \Phi_n(z) \,, \\ &\text{-------------------} \\ \rho' &= \bar{\rho}D_n Z_n(\phi)\ \ R_n(z) \,, \\ w' &= -i\nu_n D_n Z_n(\phi)\ W_n(z) \,, \\ T' &= \bar{T}D_n Z_n(\phi)\ T_n(z) \,. \end{aligned}\right\} e^{i(s\lambda-\nu_n t)} \,, \qquad (5.8)$$

where s is the longitudinal wavenumber, ν_n is the frequency and subscript n refers to a vertical mode.

Functions $U_n(\phi)$, $V_n(\phi)$ and $Z_n(\phi)$ depend on latitude only and are obtained as the meridional structure functions of the system of shallow-water equations (3.7). See Kasahara (1976) for more details.

6. Expansion of three-dimensional data

We discuss here the expansion of three-dimensional data in terms of the elementary solutions (5.8). Input data are

$$\left.\begin{aligned} u_i(\lambda,\phi,z) &= u_{obs} , \\ v_i(\lambda,\phi,z) &= v_{obs} , \\ p_i(\lambda,\phi,z) &= p_{obs} - \overline{p}(z) . \end{aligned}\right\} \tag{6.1}$$

The right-hand sides of (6.1) are observed fields of velocity components and the deviation of pressure from the basic state $\overline{p}(z)$.

By defining the input vector $\mathbb{x}_i$ and the scaling matrix S_n,

$$\mathbb{x}_i = \begin{pmatrix} u_i \\ v_i \\ p_i \end{pmatrix} , \qquad S_n = \begin{pmatrix} (gD_n)^{\frac{1}{2}} & 0 & 0 \\ 0 & (gD_n)^{\frac{1}{2}} & 0 \\ 0 & 0 & g\overline{\rho}D_n \end{pmatrix} , \tag{6.2}$$

we can express the vertical expansion of $\mathbb{x}_i$ as:

$$\mathbb{x}_i(\lambda,\phi,z) = \sum_{n=0}^{N} S_n \mathbb{X}_n(\lambda,\phi)\ \Phi_n(z) , \tag{6.3}$$

where $\mathbb{X}_n(\lambda,\phi)$ denotes the coefficients of vertical expansion.

Multiplying (6.3) by $\Phi_m(z)$ and integrating the resulting equation with respect to z from 0 to z_T, we find that

$$\int_0^{z_T} \mathbb{x}_i \Phi_m(z)\ dz = \sum_{n=0}^{N} S_n \mathbb{X}_n(\lambda,\phi) \int_0^{z_T} \Phi_m(z)\ \Phi_n(z)\ dz \ ,$$

$$\text{for } m = 0,1,\ldots, N \ . \quad (6.4)$$

This is the system of N independent equations for N unknowns $\mathbb{X}_n$. Hence, $\mathbb{X}_n(\lambda,\phi)$ are determined from $\mathbb{x}_i$ for different vertical mode n.

Once $\mathbb{X}_n(\lambda,\phi)$ are obtained, we can expand $\mathbb{X}_n(\lambda,\phi)$ in terms of Hough harmonics. Hough harmonics are defined by

$$\mathbb{H}_\ell^s(\lambda,\phi;n) = \textcircled{H}_\ell^s(\phi;n)\ e^{is\lambda} \ , \quad (6.5)$$

where

$$\textcircled{H}_\ell^s(\phi;n) = \begin{pmatrix} U_\ell^s(\phi;n) \\ -iV_\ell^s(\phi;n) \\ Z_\ell^s(\phi;n) \end{pmatrix} \quad (6.6)$$

is the Hough vector function for different vertical mode n. A Hough harmonic is two-dimensional, and superscript s and subscript ℓ refer to two-dimensional modal indices, s being zonal wavenumber and ℓ meridional index. There are two kinds of wave motions--the first for gravity-inertia waves and the second for Rossby/Haurwitz-type waves. Hough vector functions for different kinds are identified by different meridional indices-- ℓ_{EG}, ℓ_{WG} and ℓ_R for eastward-propagating gravity waves (EG), westward propagating gravity waves (WG) and Rossby/Haurwitz-type waves (R).

Two-dimensional vector $\mathbb{X}_n(\lambda,\phi)$ can be expressed in terms of Hough harmonics as

$$\mathbb{X}_n(\lambda,\phi) = \sum_{\ell=0}^{L} \sum_{s=-M}^{M} X_\ell^s(n)\ \mathbb{H}_\ell^s(\lambda,\phi;n) \quad . \tag{6.7}$$

This expansion was discussed in Kasahara (1977a) where, however, the two-dimensional field is split into the zonally averaged part and the deviation from it. The deviation field was expanded in terms of Hough harmonics, but the zonally averaged part had to be expanded in Legendre functions. The reasons for this separate treatment are that the nature of the eigenfunctions corresponding to the zonally averaged shallow-water equations was not fully understood, and the previously known set of eigenfunctions was incomplete for the expansion of arbitrary zonal field for s = 0. This deficiency was recently corrected by Kasahara (1977b), where the derivation of completely orthogonal eigenfunctions of the zonally averaged shallow-water equations is discussed. Thus, we now have a complete set of orthogonal Hough harmonics including the zonal component s = 0. Combining the normalization of $\mathbb{H}_\ell^s$, we state the orthogonality as

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1} \mathbb{H}_\ell^s \cdot \left[\mathbb{H}_{\ell'}^{s'}\right]^* d\mu d\lambda = \delta_{\ell\ell'}\delta_{ss'} \quad , \tag{6.8}$$

where $\mu = \sin\phi$, the asterisk denotes the complex conjugate, and $\delta_{\ell\ell'} = 1$ if $\ell = \ell'$ and zero otherwise, and similarly for $\delta_{ss'}$.

The orthogonality of Hough vector functions is

$$\int_{-1}^{1} Ⓗ_\ell^s \cdot \left[Ⓗ_{\ell'}^s\right]^* d\mu = \int_{-1}^{1} \left(U_\ell^s U_{\ell'}^s + V_\ell^s V_{\ell'}^s + Z_\ell^s Z_{\ell'}^s\right) d\mu$$

$$= \delta_{\ell\ell'} \quad . \tag{6.9}$$

By multiplying (6.7) by the complex conjugate of Hough harmonics, integrating the resulting equation over the entire globe and utilizing the orthogonality (6.8), we obtain the expansion coefficients $X_\ell^s(n)$ as

$$X_\ell^s(n) = \frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1} \mathbb{X}_n(\lambda,\phi)\cdot\left[\mathbb{H}_\ell^s\right]^* d\mu d\lambda \ .$$

The integral on the right-hand side can be split into two transforms:

$$X_\ell^s(n) = \int_{-1}^{1} \mathbb{X}_s(\phi;n)\cdot\left[\textcircled{H}_\ell^s\right]^* d\mu \tag{6.10}$$

and

$$\mathbb{X}_s(\phi;n) = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{X}_n(\lambda,\phi)\ e^{-is\lambda}\ d\lambda \ . \tag{6.11}$$

Eq. (6.10) is the Hough transform and (6.11) is the Fourier transform. The method of calculating these transforms is discussed in Kasahara (1977a).

7. Application of normal mode expansion to forecasting

Rinne and Karhila (1975) solved a limited-area barotropic vorticity equation model using a representation in terms of empirical orthogonal functions. Kasahara (1977a,b) applied the Hough harmonic expansion approach to integrate the barotropic primitive equations over a sphere. In this section, we discuss briefly the application of three-dimensional normal mode functions (NMFs) to integrate the baroclinic primitive equations.

The prognostic equations in a baroclinic primitive equations model in (λ,ϕ,z) coordinates may be given in the form

$$\frac{\partial u}{\partial t} - 2\Omega\sin\phi v + \frac{1}{\bar{\rho}a\cos\phi}\frac{\partial p}{\partial \lambda}$$

$$= -\mathbb{V}\cdot\nabla u - w\frac{\partial u}{\partial z} + \frac{uv}{a}\tan\phi + \frac{\rho'}{\bar{\rho}^2 a\cos\phi}\frac{\partial p}{\partial \lambda} + F_\lambda \, , \qquad (7.1)$$

$$\frac{\partial v}{\partial t} + 2\Omega\sin\phi u + \frac{1}{\bar{\rho}a}\frac{\partial p}{\partial \phi}$$

$$= -\mathbb{V}\cdot\nabla v - w\frac{\partial v}{\partial z} - \frac{u^2}{a}\tan\phi + \frac{\rho'}{\bar{\rho}^2 a}\frac{\partial p}{\partial \phi} + F_\phi \, , \qquad (7.2)$$

$$\frac{\partial p}{\partial t} - g\bar{\rho}w + \bar{p}\left(\nabla\cdot\mathbb{V} + \frac{\partial w}{\partial z}\right)$$

$$= -\mathbb{V}\cdot\nabla p - w\frac{\partial p'}{\partial z} - p'\left(\nabla\cdot\mathbb{V} + \frac{\partial w}{\partial z}\right) + \rho(\gamma - 1)\, Q \, . \qquad (7.3)$$

The diagnostic variables w, ρ and T are determined from u, v and p through the equations of vertical motion (Richardson's equation), hydrostatic equilibrium and state, respectively.

The left-hand sides of (7.1)-(7.3) are the linear parts of the prognostic equations which coincide with the corresponding prognostic equations (2.1)-(2.5). The right-hand sides of (7.1)-(7.3) are the nonlinear parts including frictional terms F_λ and F_ϕ and the heating/cooling term Q.

By defining the vector variable

$$\mathbb{W} = \begin{pmatrix} u \\ v \\ p \end{pmatrix} \, , \qquad (7.4)$$

we express Eqs. (7.1)-(7.3) by

$$\frac{\partial \mathbb{W}}{\partial t} + i\nu_n \mathbb{W} = i\,\mathbb{F}(\lambda,\phi,z,t) \quad , \tag{7.5}$$

where $i\,\mathbb{F}$ denote the vector whose components are the right-hand sides of (7.1)-(7.3). The linear parts of (7.1)-(7.3) are expressible in terms of elementary solutions (5.8), and ν_n is the normal mode frequency.

We assume that the solution of nonlinear equation (7.5) can be expressed by a series of NMFs in the form

$$\mathbb{W}(\lambda,\phi,z,t) = \sum_{n=0}^{N} \sum_{\ell=0}^{L} \sum_{s=-M}^{M} S_n C_\ell^s(t;n)\ \mathbb{H}_\ell^s(\lambda,\phi;n)\ \Phi_n(z) \ . \tag{7.6}$$

The diagnostic variables ρ', w' and T' are expressed by

$$\begin{pmatrix} \rho' \\ w' \\ T' \end{pmatrix} = \sum_{n=0}^{N} \sum_{\ell=0}^{L} \sum_{s=-M}^{M} C_\ell^s(t;n) \begin{pmatrix} \bar{\rho} & D_n R_n(z) \\ -i & \nu_\ell^s(n)\ W_n(z) \\ \bar{T} & D_n \ \ T_n(z) \end{pmatrix} Z_\ell^s(\phi;n)\ e^{is\lambda} \ , \tag{7.7}$$

where ν_n and $Z_n(\phi)$ in (5.8) are not only functions of mode n, but also of s and ℓ, so that we write $\nu_\ell^s(n)$ and $Z_\ell^s(\phi;n)$ instead.

By substituting (7.6) into (7.5), multiplying by $\left[\mathbb{H}_\ell^{s'}\right]^* \Phi_m$, integrating the resulting equation over the entire globe and through the depth 0 to z_T and utilizing the orthogonality condition (6.8), we obtain the spectral equation:

$$\sum_{n=0}^{N} s_n \alpha_{nm} \frac{dC_\ell^s(t;n)}{dt} + i \sum_{n=0}^{N} \nu_\ell^s(n)\; s_n \alpha_{nm} C_\ell^s(t;n)$$

$$= \frac{i}{2\pi} \int_0^{z_T} \int_0^{2\pi} \int_{-1}^{1} \mathbb{F}(\lambda,\phi,z,t) \cdot \left[\mathbb{H}_\ell^s(\lambda,\phi;n)\right]^* \Phi_m(z)\; d\mu d\lambda dz \;,$$

$$\text{for } m = 0,1,\ldots, N \;, \qquad (7.8)$$

where

$$\alpha_{nm} = \int_0^{z_T} \Phi_n(z)\; \Phi_m(z)\; dz \;.$$

The N unknowns $dC_\ell^s(t;n)/dt$ will be solved from the system of N simultaneous equations (7.7), and $C_\ell^s(t + \Delta t;n)$--the expansion coefficients after a time increment Δt--will be extrapolated from those at time t using finite differencing. Once $C_\ell^s(t + \Delta t;n)$ are obtained, the variable fields at time $t + \Delta t$ will be constructed by the series (7.6) and (7.7).

7. Conclusions

We presented the formulation of NMFs of a three-dimensional atmospheric model and the application of NMFs to analysis and prediction of the three-dimensional global atmosphere. A unique aspect of NMF expansion is that the analysis will demonstrate the partition of energy into two distinct kinds of motions--the first for gravity-inertia waves and the second for Rossby/Haurwitz-type waves. Both kinds of motions are also classified into different vertical modes (external and internal). Such spectral distributions of energy from the analysis will provide a sound basis to determine a necessary resolution in the representation of atmospheric data.

The NMFs discussed in this note are based on an isothermal atmosphere at rest. This is the simplest case to be considered in the hierarchy of three-dimensional atmospheric models and it offers an easier understanding of the structure of NMFs. For example, the external vertical mode characterized by no vertical motion is clearly distinguished from the internal modes.

The assumption of an isothermal atmosphere is, however, too restrictive. In fact, the structure of NMFs in the horizontal and vertical directions can be separated even in a non-isothermal atmosphere as long as the basic state remains at rest. It is planned to formulate analytical NMFs for the basic atmosphere at rest with the temperature distribution as a function of height only.

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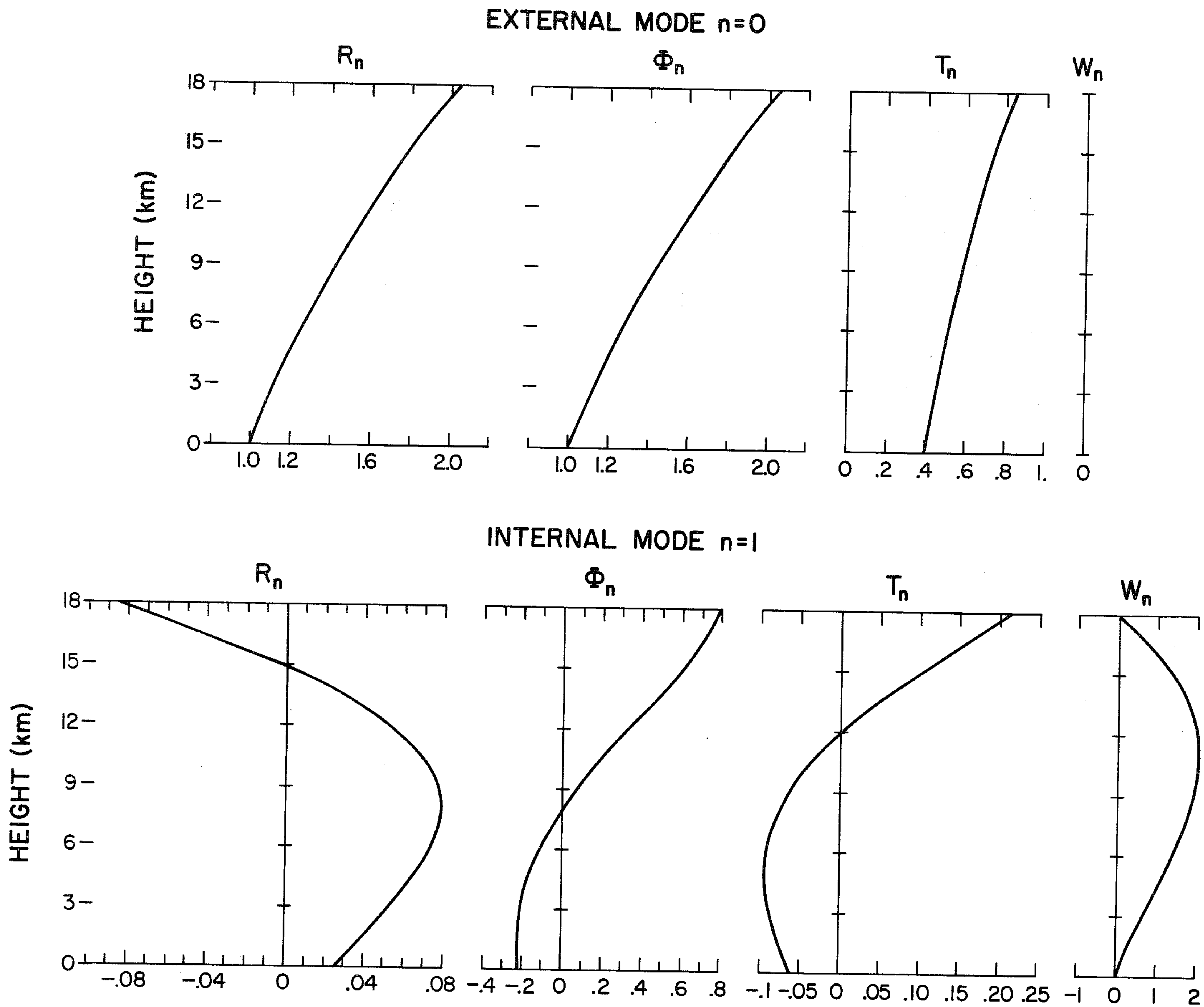
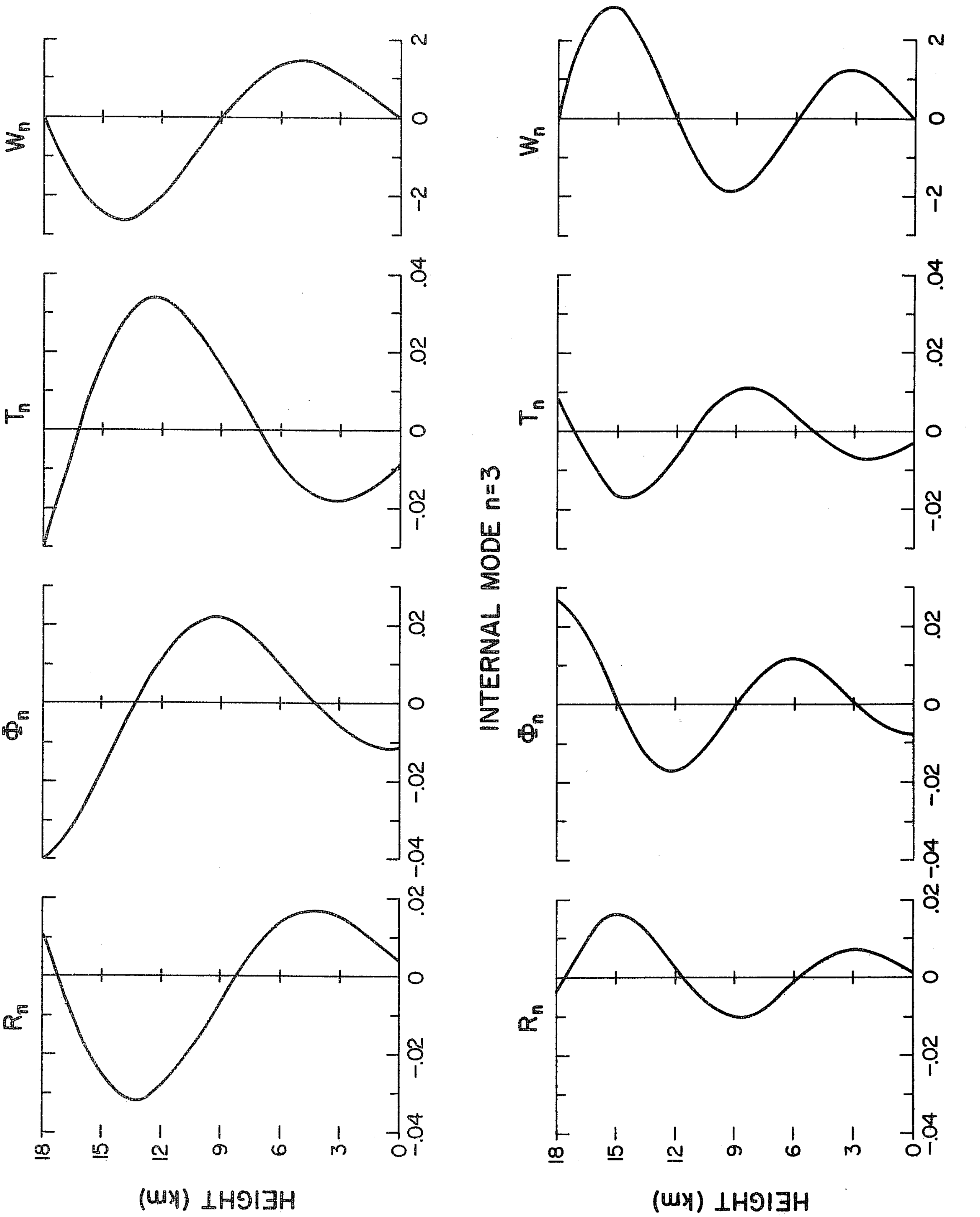
TABLE 1

1	2	3	4	5	6	7
n	Z (KM)	D_n (M)	D_n^* (M)	D_n^{**} (M)	$\sqrt{gD_n}$ (M/S)	$\varepsilon = \frac{4a^2\Omega^2}{gD_n}$
0	∞	10000	9525	9717	313	8.75
1	18.0	1131	823	1162	105	77.34
2	9.0	316	215	321	56	277.2
3	6.0	143	115	167	37	610.3
4	4.5	81	68	119	28	1076.7
5	3.6	52	31	68	22	1676.3

1. $n = 0$ EXTERNAL MODE, $n > 0$ INTERNAL MODE
2. VERTICAL SCALE
3. EQUIVALENT HEIGHT FOR AN ISOTHERMAL ATMOSPHERE
4. EQUIVALENT HEIGHT FOR REALISTIC TEMPERATURE DISTRIBUTION WITH THE TOP AT 18 KM
5. EQUIVALENT HEIGHT FOR REALISTIC TEMPERATURE DISTRIBUTION WITH THE TOP AT 36 KM
6. GRAVITY WAVE SPEED
7. LAMB'S PARAMETER



Figs. 1 and 2 (next page) illustrate the vertical structures of external mode (n = 0) and internal modes ($n \geqslant 1$) for density $R_n(z)$, geopotential $\Phi_n(z)$, temperature $T_n(z)$ and vertical motion $W_n(z)$


INTERNAL MODE n=3
HEIGHT (km)
HEIGHT (km)
Rn
Φn
Tn
Wn
Rn
Φn
Tn
Wn
18
15
12
9
6
3
0
-.04
-.02
0
.02
-.04
-.02
0
.02
-.02
0
.02
.04
-2
0
2