Toward variational data assimilation for coupled models: first experiments on a diffusion problem

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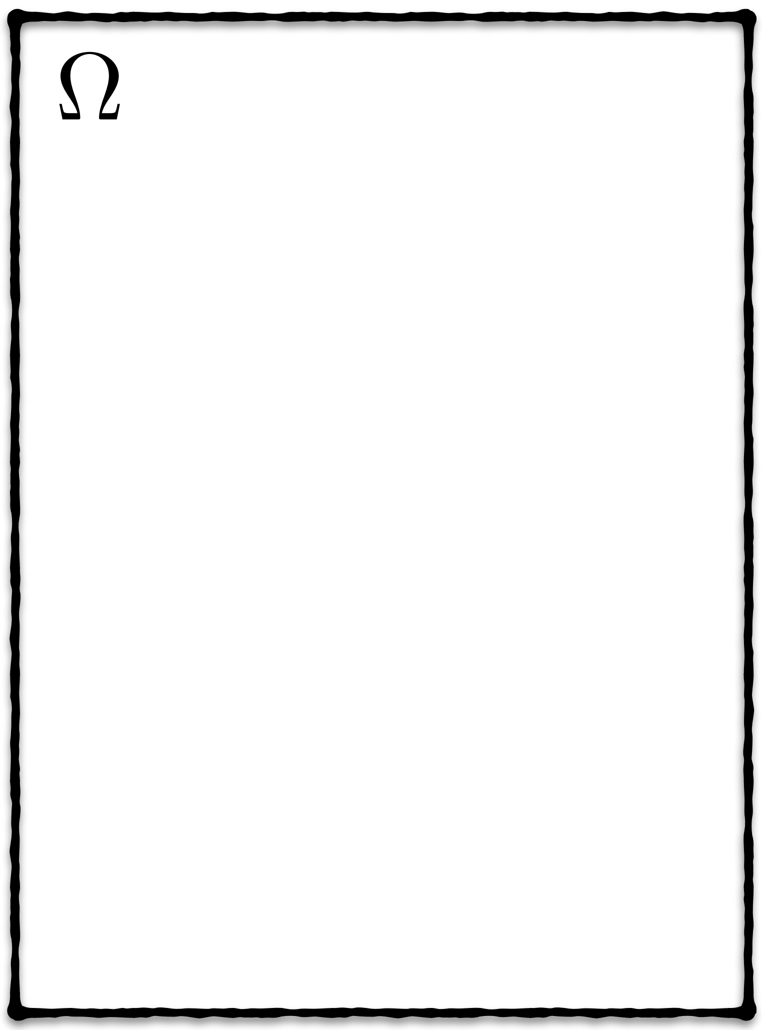
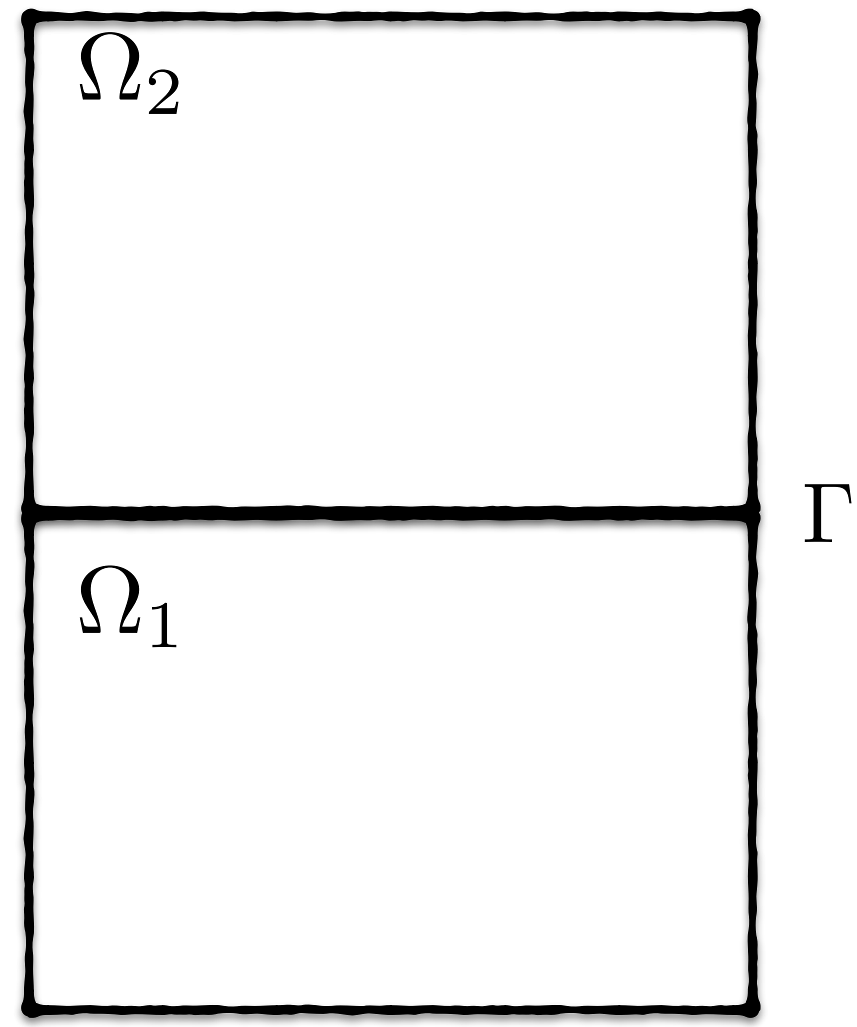
Context — Intra-model coupling

Monolithic method



- A single tridiagonal system
- Unnatural for multiphysics problems

Partitioned method



- Γ is a physical interface
- We need to deal with interface/transmission conditions

How to handle this partitioned approach in the assimilation process ?

Context — Schwarz waveform relaxation

Iterative solution of the direct partitioned problem (strong coupling)

$$\left\{\begin{array}{rcll} \mathcal{L}_1\mathbf{u}_1^k & = & f, & \text{in } \Omega_1 \times [0,T] \\ \mathbf{u}_1^k(z,0) & = & \mathbf{u}_0(z) & \text{in } \Omega_1 \\ \mathcal{F}_1\mathbf{u}_1^k & = & \mathcal{F}_2\mathbf{u}_2^{k-1} & \text{on } \Gamma \times [0,T] \end{array}\right. \qquad \left\{\begin{array}{rcll} \mathcal{L}_2\mathbf{u}_2^k & = & f, & \text{in } \Omega_2 \times [0,T] \\ \mathbf{u}_2^k(z,0) & = & \mathbf{u}_0(z) & \text{in } \Omega_2 \\ \mathcal{G}_2\mathbf{u}_2^k & = & \mathcal{G}_1\mathbf{u}_1^{k-1} & \text{on } \Gamma \times [0,T] \end{array}\right.$$

k is the iteration number

at convergence: $\mathcal{F}_1\mathbf{u}_1 = \mathcal{F}_2\mathbf{u}_2$ and $\mathcal{G}_1\mathbf{u}_1 = \mathcal{G}_2\mathbf{u}_2$ on $\Gamma \times [0,T]$

$\Rightarrow$ $\mathcal{F}_j$ and $\mathcal{G}_j$ are interface operators chosen to ensure that the coupled problem is well-posed (and possibly to accelerate convergence)

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In the context of OA coupling :

- One single iteration is performed
- Interface operators chosen to satisfy flux continuity

When assimilating data, how to combine Schwarz iterations and minimisation iterations ?

Cost function and direct model

Monolithic spirit

$$\text{let } \mathbf{x}_0 = \mathbf{u}_0(z),\ z \in \Omega = \Omega_1 \bigcup \Omega_2$$

algo 1

$$J(\mathbf{x}_0) = \|\mathbf{x}_0 - \mathbf{x}^b\|^2_{\mathbf{B}} + \sum_i \|\mathbf{y}_i - H_i(M^\star(t_i, \mathbf{x}_0, \mathbf{u}_j^0))\|^2_{\mathbf{R}_i}$$

$M^\star(t_i, \mathbf{x}_0, \mathbf{u}_j^0) =$ converged solution of the Schwarz algorithm at time t_i, with IC $\mathbf{x}_0$ and first guess $\mathbf{u}_j^0$ (j=1,2)

We need the adjoint of the coupled system and the iterative scheme

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Could the minimization iterations compensate for coupling iterations ?

algo 2

$$J(\mathbf{x}_0) = \|\mathbf{x}_0 - \mathbf{x}^b\|^2_{\mathbf{B}} + \sum_i \|\mathbf{y}_i - H_i(M^\dagger(t_i, \mathbf{x}_0, \mathbf{u}_j^0))\|^2_{\mathbf{R}_i}$$

$M^\dagger(t_i, \mathbf{x}_0, \mathbf{u}_j^0)$ = truncated Schwarz algorithm (stopped after a fixed number of iterations)

We need the adjoint of the coupled system and the iterative scheme

Cost function and direct model

Partitioned spirit

$$\text{Let } \mathbf{x}_0 = (\mathbf{u}_0(z), \mathbf{u}_1^0(0,t), \mathbf{u}_2^0(0,t))^T$$

Additional term in the previous cost function to penalize the mismatch in the interface conditions

$$J_{\text{int}} = \|\mathcal{F}_1\mathbf{u}_1 - \mathcal{F}_2\mathbf{u}_2\|_{\mathbf{F}}^2 + \|\mathcal{G}_1\mathbf{u}_1 - \mathcal{G}_2\mathbf{u}_2\|_{\mathbf{G}}^2$$

algo 3

$$J(\mathbf{x}_0) = \|\mathbf{x}_0 - \mathbf{x}^b\|_{\mathbf{B}}^2 + \sum_i \|\mathbf{y}_i - H_i(M^\dagger(t_i, \mathbf{x}_0, \mathbf{u}_j^0))\|_{\mathbf{R}_i}^2 + \alpha J_{\text{int}}$$

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algo 4

$$J(\mathbf{x}_0) = \|\mathbf{x}_0 - \mathbf{x}^b\|_{\mathbf{B}}^2 + \sum_i \|\mathbf{y}_i - H_i(M^0(t_i, \mathbf{x}_0, \mathbf{u}_j^0))\|_{\mathbf{R}_i}^2 + \alpha J_{\text{int}}$$

$M^0(t_i, \mathbf{x}_0, \mathbf{u}_j^0)$ = solution after one single integration of each model taken separately

→ Remove the coupling iterations from the direct model

Algorithm	Penalization of the interface	Adjoint of the coupling	Strongly coupled solution	Number of Schwarz/ coupling iterations
Algo 1	NO	YES	YES	convergence
Algo 2	NO	YES	NO	truncated
Algo 3	YES	YES	NO	truncated
Algo 4	YES	NO	NO	1

Model problem

Linear problem

$$\mathcal{L}_j := \partial_t - \partial_z(\nu_j \partial_z)$$

$$\nu_1 \neq \nu_2$$

$$\mathcal{F}_j = \mathrm{Id}$$

$$\mathcal{G}_j = \nu_j \partial_z$$

Non-linear unstratified problem with parameterizations

Coupled SCMs

time

Numerical experiments

- Choose rhs to have an analytical solution
- Background obtained from biased initial state
- Observations are generated at the end of the time-window (in the interior, away from the interface)
- $\mathbf{R} = 1,\ \mathbf{B} = 100,\ \mathbf{F} = \mathbf{G} = 10$

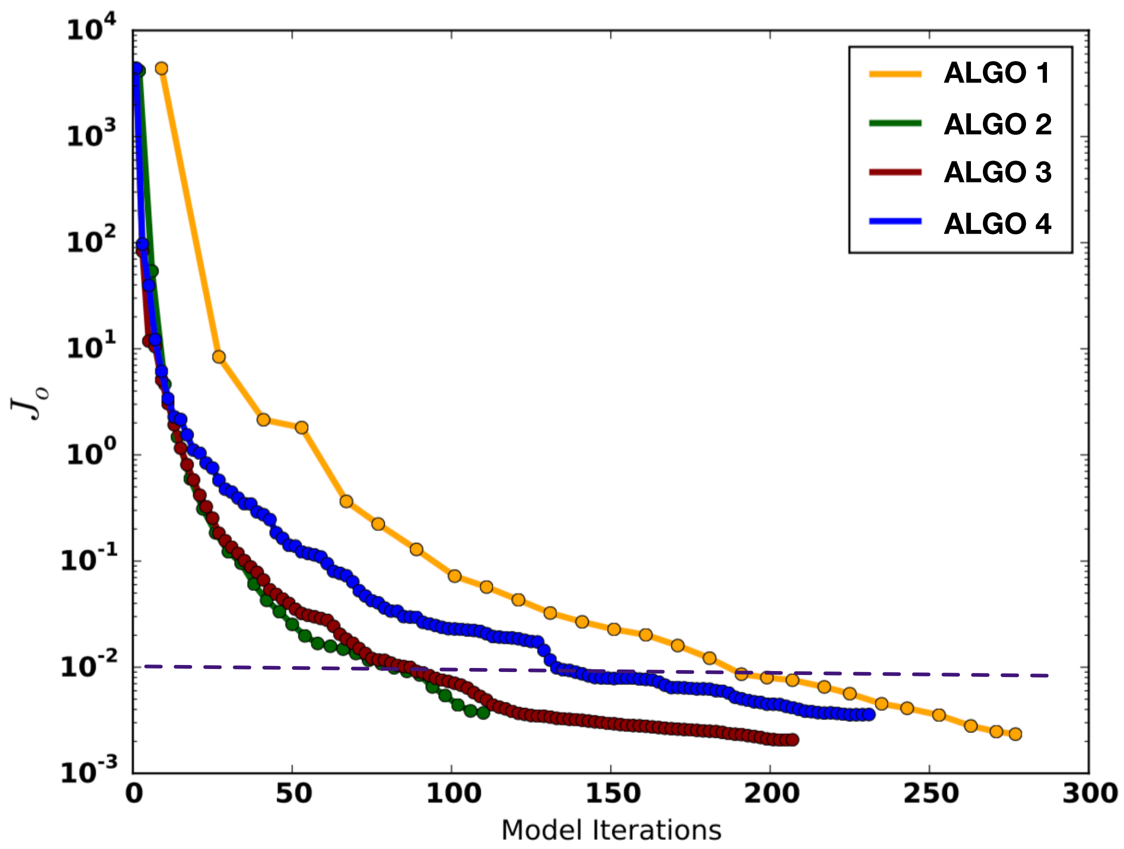

Figure 1. *Evolution of J_o with respect to the total number of iteration model (direct and adjoint). each dot represents a minimization iteration*

	Normalized RMSE	# of iterations after minimization
Algo 1	1	2
Algo 2	1.11	3
Algo 3	1.07	3
Algo 4	1.94	5

Conclusions and perspectives

- Conclude on this highly simplified linear problem

Pelleréj, R., Vidard, A., Lemarié, F : Toward variational data assimilation for coupled models: first experiments on a diffusion problem, *in preparation for CARI'16*

- Develop increasingly complex testcases within OOPS
  - add surface layer param to compute interface conditions
  - add turbulent vertical mixing param

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Thank you !

If you have any tricky question on variational data assimilation, please kindly contact arthur.vidard@inria.fr