VORTICITY TRANSPORT IN DEEP CUMULUS CONVECTION

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ABSTRACT

This paper is written in three sections. The first describes how the apparent complexity of mean vorticity production and transport in regions of deep convection may be considerably simplified using what is, in effect, a form of the circulation theorem. The second section describes the results of vorticity analyses based on this approach and the third proposes a scheme for the parameterisation of cumulus dynamics based on the vorticity equation, but which is nevertheless readily adaptable for use with the primitive equations.

1. INTRODUCTION

It has long been recognised that in the vertical heat and moisture transfer by a population of deep convective clouds, the eddy correlation terms $(\partial/\partial p(\widetilde{\omega'\theta'}), \partial/\partial p(\widetilde{\omega' q'}))$ play a crucial role, and several parameterisation schemes for these terms have been suggested. The parameterisation of the vertical momentum (or vorticity) transport has received relatively little attention although diagnostic studies using tropical data suggest that this could be highly important. One reason for lack of progress in their parameterisation is their apparent complexity, advection, stretching and twisting effects all being important. In the next section an approach based on the circulation theorem is described which leads to a relatively simple formulation of cumulus population dynamics. Some diagnostic results are then described and, in the final section, a parameterisation scheme based on these ideas is proposed.

2. SIMPLIFICATION OF CUMULUS DYNAMICS RESULTING FROM APPLICATION OF THE CIRCULATION THEOREM

The vorticity equation may be written (in pressure coordinates) as

$$\frac{\partial Z}{\partial t} = -\nabla\cdot(\underline{v}Z) - \frac{\partial}{\partial p}(\omega Z) - Z\nabla_h\cdot\underline{v} + \underline{\zeta}_h\cdot\nabla w + F \qquad (1)$$

where Z denotes the vertical component of absolute vorticity $(= f + \zeta)$, $\underline{\zeta}_h$ the horizontal component of relative vorticity, w the vertical velocity and F the contribution of sub-convection scale eddies.

Let $\sim$ denote an average over a grid area, of $(100\ \text{km})^2$, say, in an isobaric surface. The twisting term is first expressed as

$$\underline{\zeta}_h\cdot\nabla w = \nabla w\cdot\underline{k}\times\frac{\partial\underline{v}}{\partial z} = -\underline{k}\cdot\nabla\omega\times\frac{\partial\underline{v}}{\partial p}$$

so that, on averaging,

$$\widetilde{\underline{\zeta}_h\cdot\nabla w} = -\frac{1}{A}\underline{k}\cdot\int\nabla\omega\times\frac{\partial\underline{v}}{\partial p}\,dA = -k\cdot\widetilde{\nabla\omega}\times\frac{\partial\tilde{\underline{v}}}{\partial p} - \frac{1}{A}k\cdot\int(\nabla\omega)'\times\frac{\partial\underline{v}'}{\partial p}\,dA$$

where A is the averaging area and the prime denotes a deviation from the average. But $(\nabla\omega)' = \nabla\omega - \widetilde{\nabla\omega} = \nabla\omega' - \widetilde{\nabla\omega}$, so that the integral is $\frac{k}{A}\int \omega' \frac{\partial \zeta'}{\partial p} dA - \frac{k}{A}\int \nabla_h \times \left(\omega' \frac{\partial \underline{v}'}{\partial p}\right) dA$. Applying Stoke's Theorem to the second integral, it may be expressed as $\frac{k}{A}\oint \omega_c' \frac{\partial v_t'}{\partial p} d\ell$ where ω_c' is the value of ω' and v_t' the tangential component of $\underline{v}'$ on the contour c surrounding A. Letting ^ denote an average value on c, and $''$ a deviation, $\omega_c' = \omega_c'' + \hat{\omega} - \tilde{\omega}$, and, finally,

$$\widetilde{\underline{\zeta}_h \cdot \nabla w} = - k \cdot \widetilde{\nabla\omega} \times \frac{\partial \tilde{\underline{v}}}{\partial p} - \widetilde{\omega \frac{\partial Z}{\partial p}} - \hat{\omega} \frac{\partial \hat{\zeta}}{\partial p}$$

, neglecting the correlation between ω_c'' and $\frac{\partial v_t''}{\partial p}$. Thus averaging Eq. (1) gives

$$\frac{\partial \tilde{Z}}{\partial t} = - \widetilde{\nabla \cdot (\underline{v} Z)} - \hat{\omega} \frac{\partial \hat{\zeta}}{\partial p} - \underline{k} \cdot \widetilde{\nabla\omega} \times \frac{\partial \tilde{\underline{v}}}{\partial p} + \tilde{F} \qquad (2)$$

In this form of the vorticity equation, the detail of the twisting, stretching and vertical advection of vorticity on the cumulus scale is to a large extent by-passed through the device of introducing $\hat{\omega}$. The first term on the right can be expressed as $- \frac{L}{A} \widehat{v_n Z}$ where L is the length of c, i.e. as $- \frac{L}{A}(\hat{v}_n \hat{Z} + \widehat{v_n'' Z''}) = -(\widetilde{\nabla \cdot \underline{v}})\tilde{Z} + \widetilde{\nabla \cdot \underline{v}}(\tilde{Z} - \hat{Z}) - \frac{L}{A} \widehat{v_n'' Z''}$.

Thus Eq. (2) may be written as

$$R(p) = \frac{\partial \tilde{Z}}{\partial t} + (\widetilde{\nabla \cdot \underline{v}})\tilde{Z} + \underline{k} \cdot \widetilde{\nabla\omega} \times \frac{\partial \tilde{\underline{v}}}{\partial p} = - \hat{\omega} \frac{\partial \hat{\zeta}}{\partial p} + (\widetilde{\nabla \cdot \underline{v}})(\tilde{Z} - \hat{Z}) - \frac{L}{A} \widehat{v_n'' Z''} \qquad (2')$$

The distribution of ω in an isobaric surface in the mid-troposphere penetrated by deep convective cells may be idealised as shown in Fig. 1.

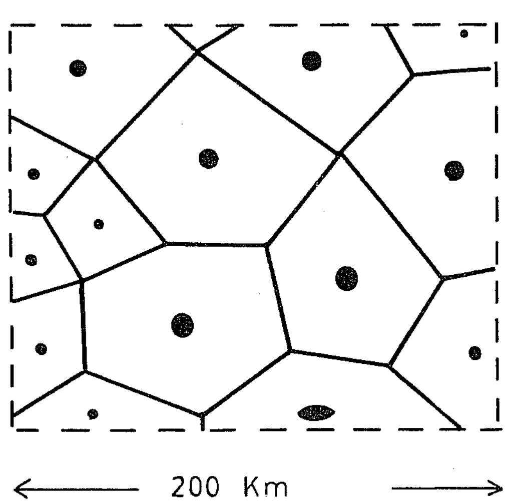

Fig. 1 Schematic distribution of vertical motion in an isobaric surface in the middle troposphere where deep convection is taking place. The dots denote regions of concentrated ascent and the lines are contours enclosing complete convective cells.

The schematic cells in Fig. 1 have their boundaries drawn such that, for each individual cell, the cell-average vertical motion ($\tilde{\omega}^c$) is equal to the grid-average ($\tilde{\omega}$). Thus for the purposes of the 'grid-area' averaging the 'grid' must be envisaged as containing a discrete number of cells. Since each cell is characterised by a concentrated ascending core completely surrounded by a (generally) descending environment, $\hat{\omega}$ for the grid area must be the value round the contour enclosing a number of complete cells, i.e. a value (> 0) typical of this environment .

The simplicity of this result derives from the fact that Eq. (2) (ignoring $\tilde{F}$) is essentially a statement of the circulation theorem applied to fluid elements on the contour c (see Appendix).

3. DIAGNOSTIC RESULTS

3.1 General results

Part of the above analysis was given by Pearce and Riehl (1969) together with a vorticity diagnosis of convective systems in the Caribbean, including some hurricane cases. The results were consistent with Eq. (2) with $\hat{\omega}$ assumed > 0, the columns selected being characterised by $\tilde{\zeta}$ decreasing with height, i.e. $\frac{\partial \tilde{\zeta}}{\partial p} > 0$, and ignoring $\widetilde{\underline{v}'\zeta'}$ correlations. The cumulus population provided a vorticity sink resulting from advection in the descending air between the clouds.

More recent analyses, e.g. those of Ruprecht and Gray (1976), Shapiro and Stevens (1980) and Chu, Yanai and Sui (1981) have indicated that the cumulus population may provide a vorticity source at some upper levels rather than a sink when $\frac{\partial \tilde{\zeta}}{\partial p} > 0$. The above theory indicates that this can arise through $\tilde{Z} - \hat{Z}$ and $\widehat{v_n''Z''}$, the correlation between the outward velocity v_n and Z' on the boundary c. It is interesting to note that in these latter cases there is a quite substantial mean shear $\frac{\partial \underline{v}}{\partial z}$ over the region considered whereas in the Caribbean cases considered by Pearce and Riehl the shear was small. The observed differences in cumulus vorticity sources may thus arise from different deep convective structures in the differing large-scale shears.

3.2 <u>An example of the computation of $-\hat{\omega}\, \partial\tilde{\zeta}/\partial p$</u>

Shapiro and Stevens (1980) carried out a detailed analysis of vorticity and vertical motion for 'category 4' of GATE Phase III composites, i.e. of the part of the composite waves having maximum convective activity. They estimated, in addition to $\tilde{\omega}$, upwards cloud mass fluxes (M_c) from the thermodynamic equations using Johnson's (1978) method with a spectral cloud model. It is thus a simple matter to compute the mean vertical motion in the cloud-free air and identify this with $\hat{\omega}$ using

$$\hat{\omega} = M_c + \tilde{\omega} \quad .$$

The resulting distribution of $\hat{\omega}(p)$, together with that of $\tilde{\zeta}(p)$, is shown in Fig. 2.

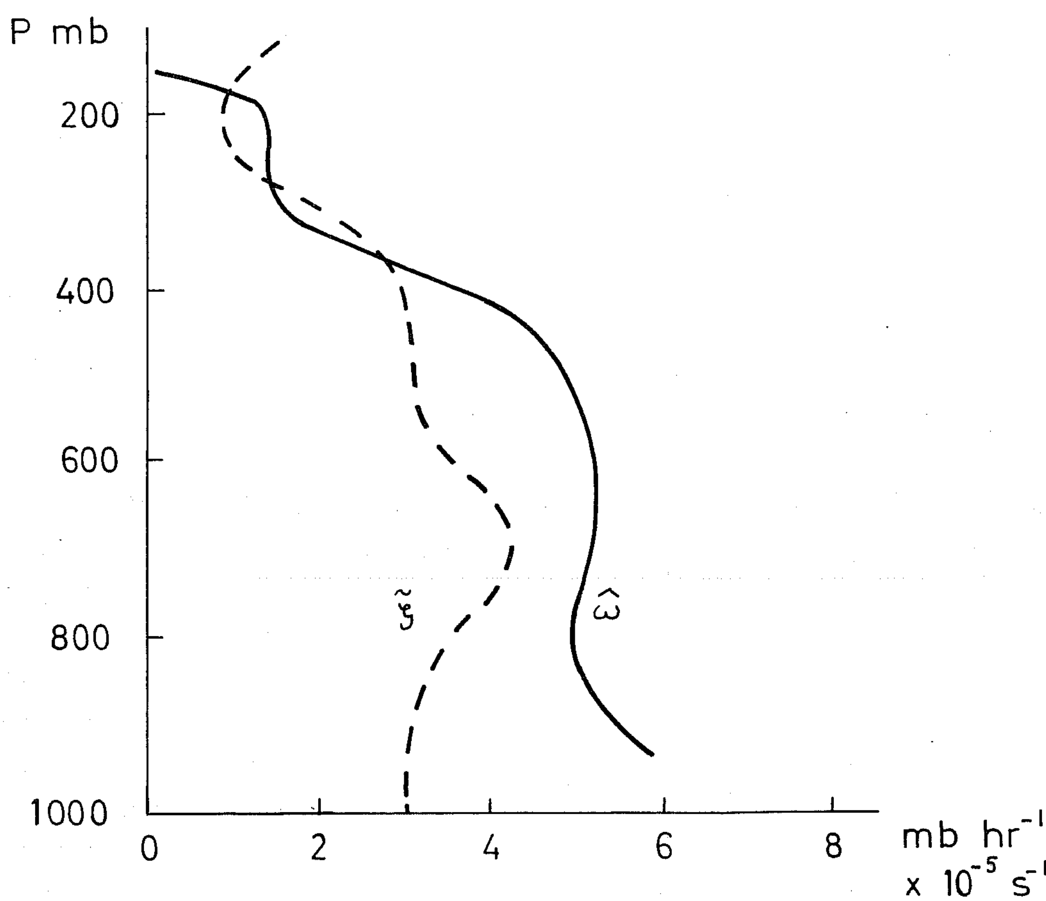

Fig. 2 Vertical distributions of $\hat{\omega}(p)$ and $\tilde{\zeta}(p)$ (dashed) for category 4 GATE Phase III (values from Shapiro and Stevens (1980)).

Using their numerical values the distribution of $-\hat{\omega}\frac{\partial\tilde{\zeta}}{\partial p}$ has been computed. This is shown in Fig. 3, together with the distribution of the 'residual' $R(p) = \frac{\partial\tilde{\zeta}}{\partial t} + \widetilde{\nabla\cdot(\underline{v}Z)} + \underline{k}\cdot\widetilde{\nabla\omega} \times \frac{\partial\tilde{\underline{v}}}{\partial p}$.

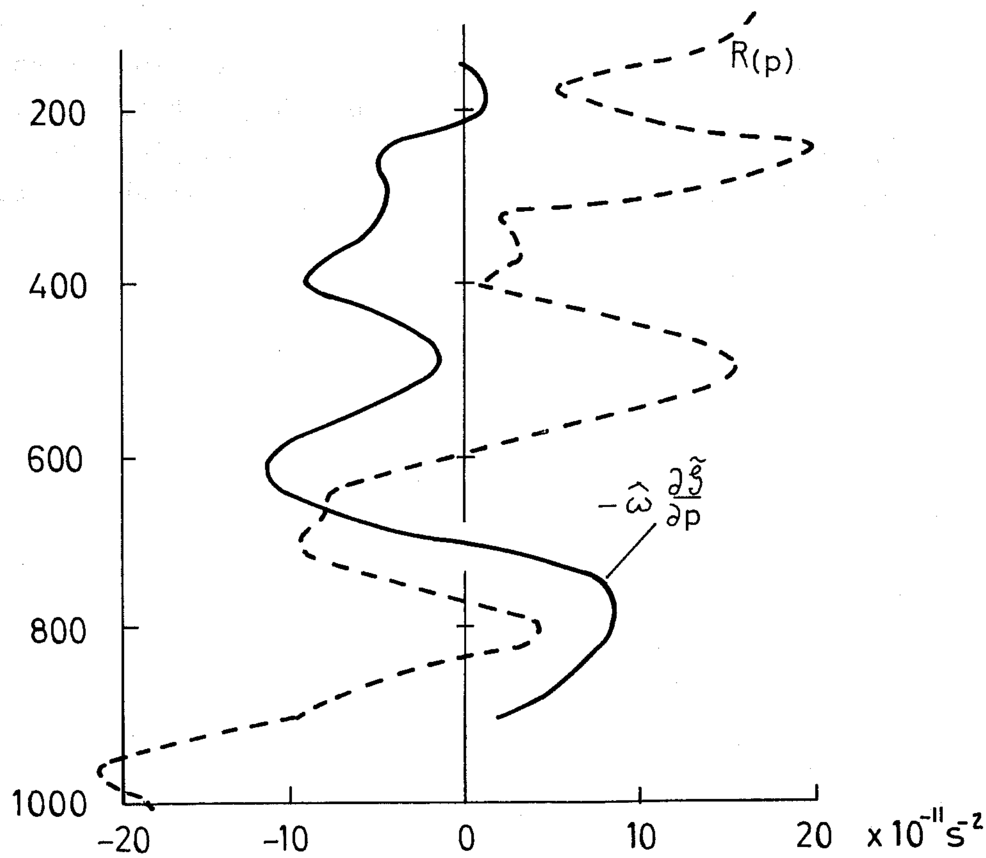

Fig. 3 Vertical distributions of $-\hat{\omega}\frac{\partial\tilde{\zeta}}{\partial p}$ (dashed, from values in Fig. 2) and R(p) (from Shapiro and Stevens (1980)).

The two curves in Fig. 3 follow each other in general form but there is a large consistent difference above about 650 mb. This may be interpreted as suggesting that in these cases a major part of the contribution of the deep convective elements to the vorticity budget is through the $\hat{\omega}$ term in Eq. (2) through most of the depth of the troposphere, but that at the main outflow levels in the mid- and upper troposphere there is, in addition, a positive contribution resulting from $\hat{Z} < \tilde{Z}$ and/or $\widehat{v_n''Z''} < 0$ (Eq. (2′)). Both are to be expected if the cumulus outflow air tends to conserve its angular momentum about the cloud axis implying that the outflow from these clouds ($v_n'' > 0$) has low vorticity. It is interesting to note the high sensitivity of $-\hat{\omega}\frac{\partial\tilde{\zeta}}{\partial p}$ to the shape of the $\tilde{\zeta}(p)$ curve which accounts for much of the variability of the R(p) distribution.

3.3 Vorticity budgets of systems traversing the GATE A-scale area

Amador (1981) computed each of the terms in the vorticity budget every twelve hours for 4 wave systems crossing the GATE area during Phase III. The budgets were for an 8° × 8° area centred on the system at each time and refer to the whole depth of the troposphere (1000 - 100 mb). (Computations were also carried out for a 4° × 4° area and gave similar results.) The main results for the first of the systems, which travelled from 19°W, 11°N on 2nd September to 69°W 21°N on 10th September, are shown in Fig. 4.

Only the main terms in Eq. (2) are shown - the surface friction contribution (F) was also estimated using the usual aerodynamic formula and the large-scale value of $-\hat{\omega}\frac{\partial\bar{\tilde{\zeta}}}{\partial p}$ (the ―― denotes a vertical average), but both of these terms are generally small, with maxima of about 0.2×10^{-10} s^{-2}. The beta-effect is generally of the same order as the horizontal advection of relative vorticity in $\widetilde{\nabla\cdot(\underline{v}\zeta)}$ (denoted by $\bar{D}$).

The residual is seen to be of the same order of magnitude as $\frac{\partial\bar{\tilde{\zeta}}}{\partial t}$ and $\bar{D}$. It must be interpreted as primarily the cumulus-scale contributions to $-\overline{\hat{\omega}\frac{\partial\tilde{\zeta}}{\partial p}}$ and $-\overline{\widetilde{\nabla\cdot(\underline{v}'\zeta')}}$ (denoted by $\overline{D'}$) plus, of course, error. The errors are difficult to estimate, but since neither of the two main terms involves the vertical motion these errors are unlikely to dominate the residual. If this is the case, then the result implies that $\bar{\tilde{\zeta}}(t)$ is determined primarily by $\bar{D}$ and the cumulus dynamics, at least from 4th September when $\tilde{\omega} < 0$. The general decrease of $\bar{\tilde{\zeta}}$ until 6th September results largely from the negative contribution of the residual. This was more than offset by positive $\bar{D}$ (mainly the beta-effect) during the latter part of the period.

A further case study is shown in Fig. 5. This system became hurricane Fifi two to three days beyond the period shown.

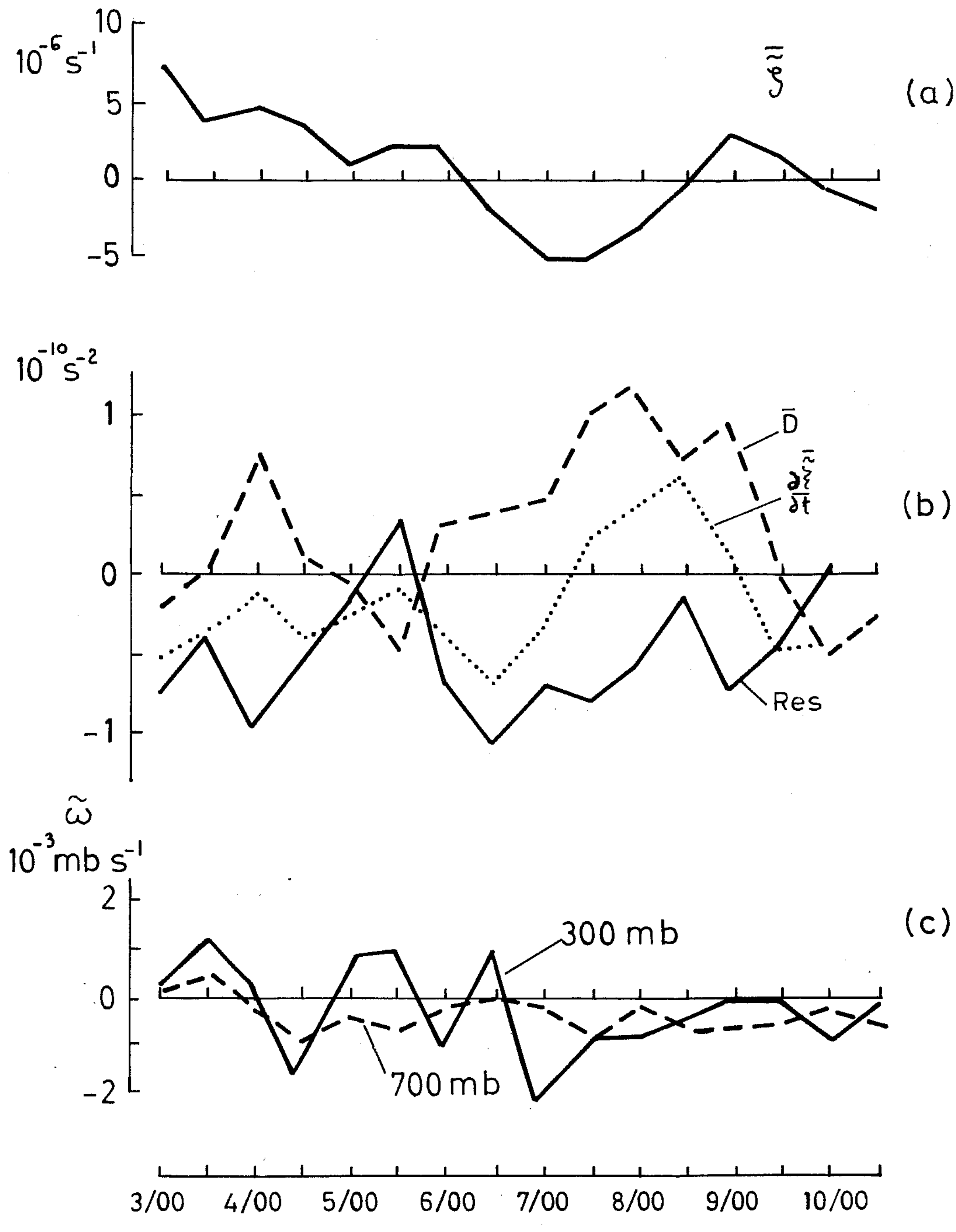

Fig. 4 Vorticity budget and vertical motion over an 8°×8° area centred on a system (Case 1) crossing the Atlantic during Phase III of GATE; (a) Mean vorticity from 1000 mb to 100 mb, (b) the individual budget components $\partial\bar{\zeta}/\partial t$ (dotted). $\bar{D}$ (dashed) and the residual (full curve), and (c) $\tilde{\omega}$ at 300 mb (full curve) and 700 mb (dashed). The computed values are at 12 hr intervals.

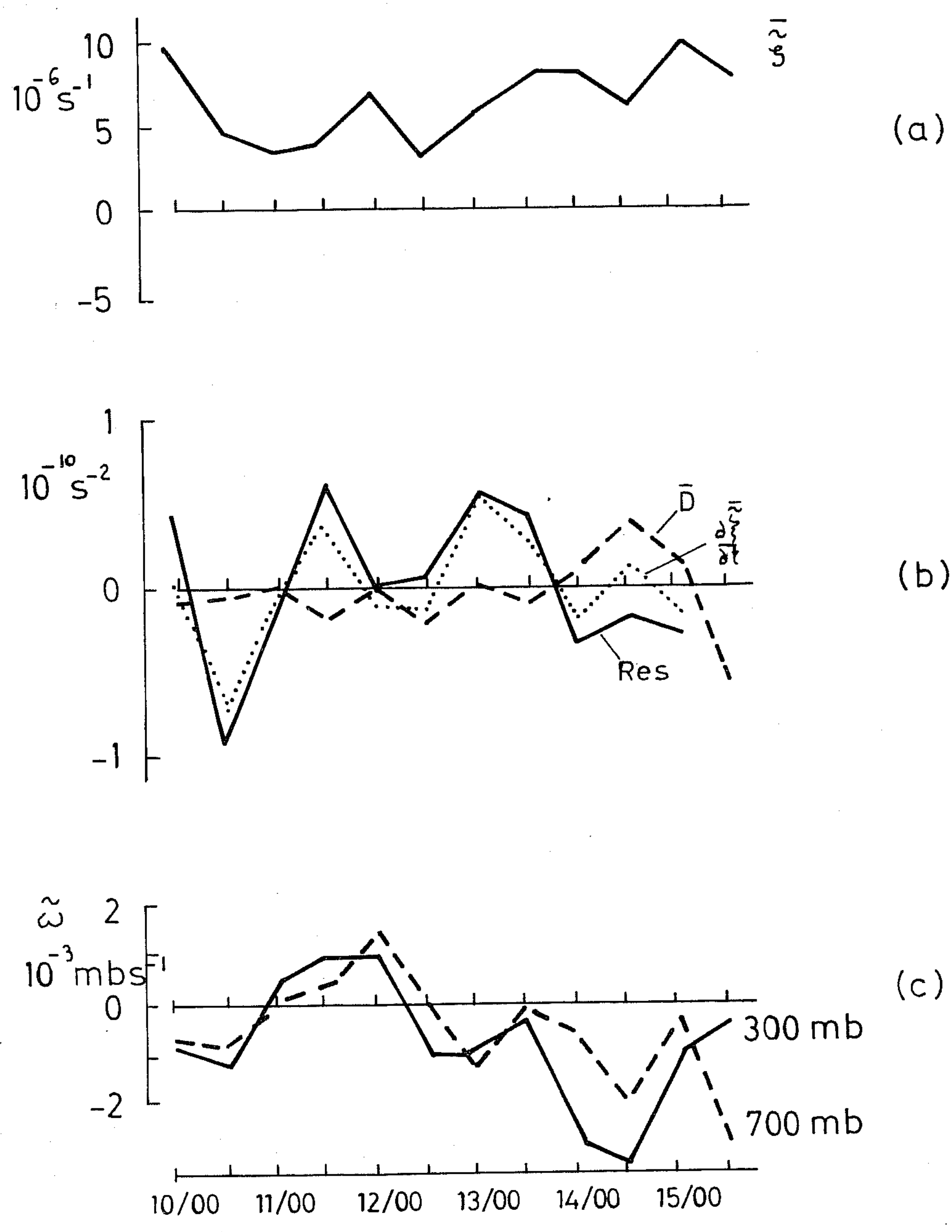

Fig. 5 As for Fig. 4, but for Case 2.

This contrasts with Case 1 in two main respects. The first is that $\bar{\tilde{\zeta}}$ does not fall to as low a value before increasing because $\bar{D}$ is generally small for most of the period shown, $\bar{\tilde{\zeta}}(t)$ being determined mainly by the cumulus dynamics. The second feature which differs from Case 1 is that, from 12th - 14th September, when $\tilde{\omega} < 0$ and cumulus activity can be assumed to be present, the contribution of the cumuli to $\frac{\partial \bar{\tilde{\zeta}}}{\partial t}$ is persistently positive, and $\bar{\tilde{\zeta}}$ is maintained at a much higher value than in Case 1 as a result of this.

3.4 Dynamical interpretation of diagnostic studies

In all of the studies referred to above the errors could be quite large and conclusions can only be tentative. However it seems clear that the contribution of the cumulus dynamics to the large-scale vorticity budget is large and can be either positive or negative. The theoretical framework described by Eq. (2) suggests that its sign depends at all levels, apart from those where outflow occurs, on the sign of $\frac{\partial \tilde{\zeta}}{\partial p}$ ($\hat{\omega}$ being assumed positive); at outflow levels it is generally positive.

The latter must result from the cumulus outflow air having a vorticity which is lower than that of the air already at that level, and which may be generally converging there. This is plausible if the cumulus air diverges sufficiently rapidly, even though it will have converged at lower levels and acquired large vorticity there. This effect may be even more significant in ensuring $\tilde{Z} - \hat{Z} > 0$ in Eq. (2´). The above studies suggest that the main cumulus outflow may occur either just below the tropopause or in the middle troposphere (Fig. 3 and Figs. 4 and 5 where $\tilde{\omega} > 0$ at 300 mb and < 0 at 700 mb).

4. PARAMETERISATION OF CUMULUS VORTICITY (OR MOMENTUM) TRANSPORT IN THE TROPICS

In primitive equation grid-point models it is obviously more convenient numerically to parameterise momentum transport by cumulus clouds rather than vorticity transport. In spectral models which integrate the vorticity and divergence equations it is more convenient to parameterise the vorticity. Neither vorticity nor momentum is conserved so that there is no obvious fundamental distinction. However, the simple form of Eq. (2) and the diagnostic studies described above suggest that a parameterisation scheme based on vorticity might provide a realistic representation of cumulus dynamics if (a) the main outflow level, (b) $D'(p)$ and (c) $\hat{\omega}(p)$ are specified. [A vorticity representation can be used in grid point p.e. models, but in this case the whole distribution of cumulus vorticity increments in each isobaric surface must be used to infer the momentum increments by solving a Poisson equation.]

The following scheme is suggested for experimental purposes:

1. Select $\tilde{\omega}$ at 700 mb ($= \tilde{\omega}_7$) and $\tilde{\omega}$ at 300 mb ($= \tilde{\omega}_3$). Specify parameters ω_{c1} , ω_{c2} (50 and -50 mb day^{-1}, say).

2. Test. (a) <u>If $\tilde{\omega}_7 > \omega_{c1}$</u> then put $\hat{\omega}(p) = \tilde{\omega}(p)$, $D'(p) = 0$ at all levels. This assumes that there is no deep cumulus activity in the grid volume.

   (b) <u>If $\omega_{c1} > \tilde{\omega}_7 > \omega_{c2}$</u>

   (i) If $\tilde{\omega}_3 > 0$ assume middle-level outflow with $\hat{\omega}(p)$ and $D'(p)$ of the forms shown in Fig. 6(a).

   (ii) If $\tilde{\omega}_3 < 0$ assume upper tropospheric outflow with $\tilde{\omega}(p)$ and $D'(p)$ as shown in Fig. 6(b).

   (c) <u>If $\tilde{\omega}_7 < \omega_{c2}$</u>

   As case (b), but with $\omega'(p)$ and $D'(p)$ as shown in Figs. 6(c) and 6(d), i.e. representing more vigorous convection.

[$D'(p)$ is the equivalent of $(\widetilde{\nabla \cdot \underline{v}})(\tilde{Z} - \hat{Z}) - \frac{L}{A} \widehat{v_n''Z''}$ in Eq. (2').]

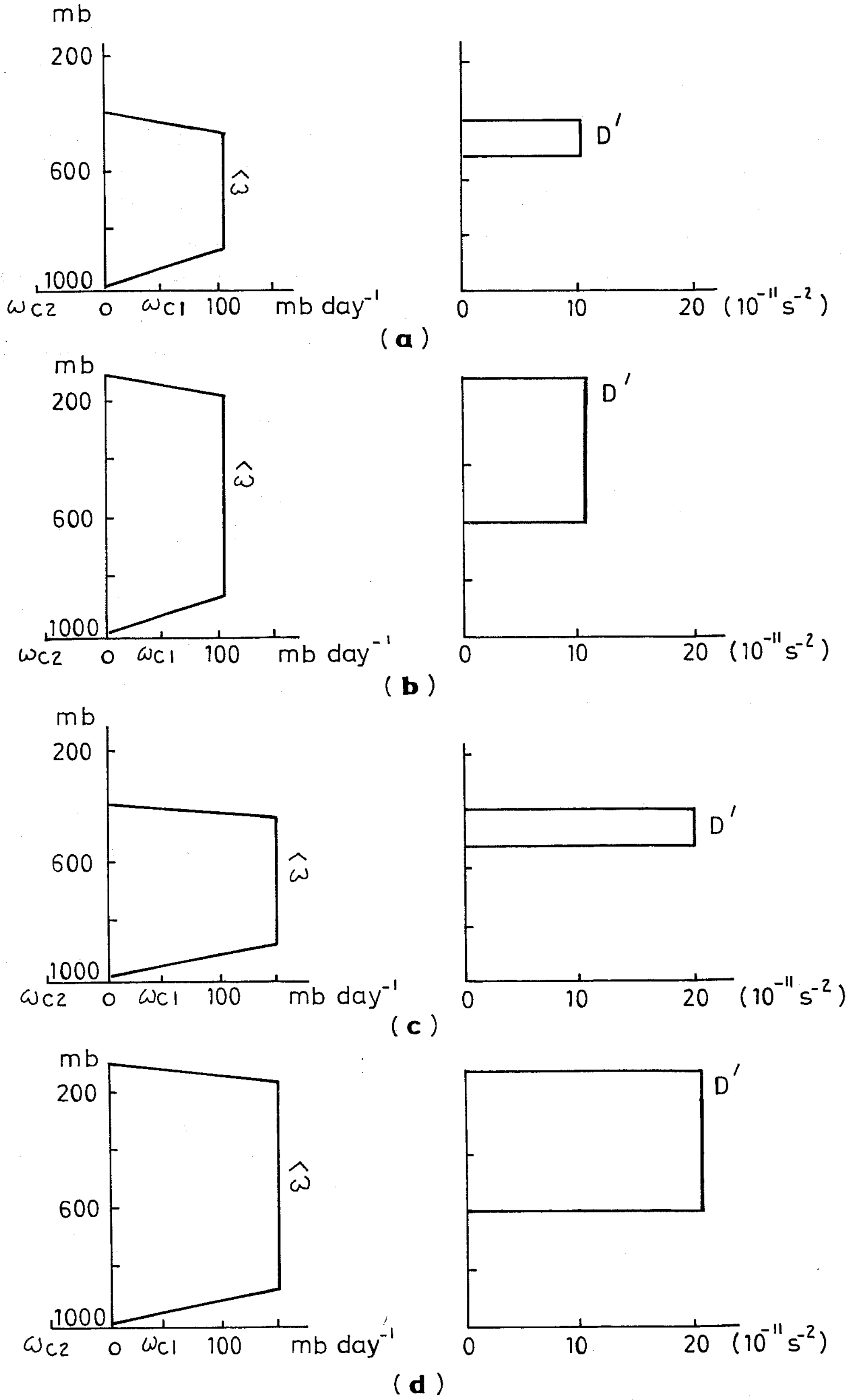

Fig. 6 Suggested profiles of $\hat{\omega}(p)$ and $D'(p)$ based on diagnostics of tropical convective activity: (a),(c) for middle layer outflow; (b),(d) for upper level outflow.

An advantage of using a scheme of this kind is that specification of $\hat{\omega}(p)$ and, by implication, the cloud mass flux M_c, enables the cumulus heat and moisture transfers also to be parameterised completely consistently by expressing $\frac{\partial}{\partial p}(\widetilde{\omega'\theta'})$ and $\frac{\partial}{\partial p}(\widetilde{\omega'q'})$ in terms of $\hat{\omega}(p)$ in the thermodynamic and moisture equations. Such a scheme has been used in the latter contexts at Reading University.

Refinement of vorticity or momentum parameterisation schemes must await a fuller understanding of the structures of the variety of cumulus ensembles in the tropics from further diagnostic and numerical modelling studies.

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APPENDIX - *EQUIVALENCE OF EQ. (2) AND THE CIRCULATION THEOREM*

The circulation C round the contour c enclosing a plane area A is defined by

$$C\,\underline{k} = \underline{k}\oint_c \underline{v}\cdot d\underline{\ell}$$

where $\underline{k}$ is unit vector normal to the plane.

The rate of change of the circulation following the fluid elements is given by

$$\underline{k}\,\frac{D_f C}{Dt} = \underline{k}\oint_c \frac{D\underline{v}}{Dt}\cdot d\underline{\ell} + \underline{k}\oint_c \underline{v}\cdot(d\underline{\ell}\cdot\nabla)\underline{v} \quad .$$

The second integral is zero since the integrand may be expressed as $d\left(\frac{v^2}{2}\right)$. Writing $\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}_h\cdot\nabla)\underline{v}_h + \omega\,\frac{\partial \underline{v}}{\partial p} + f\underline{k}\times\underline{v}$, and noting $(\underline{v}_h\cdot\nabla)\underline{v}_h = \nabla\left(\frac{v_h{}^2}{2}\right) - \underline{v}_h\times\zeta\underline{k}$, gives (with $\underline{k}$ taken vertically upwards)

$$\underline{k}\,\frac{D_f C}{Dt} = \underline{k}\,\frac{\partial C}{\partial t} + \underline{k}\,\hat{\omega}\,\frac{\partial C}{\partial p} - \underline{k}\oint_c \underline{v}_h\times Z\underline{k}\cdot d\underline{\ell} - A\underline{k}\cdot\widetilde{\nabla\omega}\times\frac{\partial\tilde{\underline{v}}}{\partial p} \qquad (A1)$$

where $\hat{\omega}$ is defined as the average of ω $(=\hat{\omega}+\omega_c'')$ on c and ω_c'' is assumed to be uncorrelated with $\frac{\partial v_t}{\partial p}$ (the tangential component of $\frac{\partial \underline{v}}{\partial p}$ on c).

The integral may be expressed as $-\,\underline{k}\int_A \nabla\times(\underline{v}_h\times Z\underline{k})\,dA$.

But $\nabla\times(\underline{v}_h\times Z\underline{k}) = Z\,\frac{\partial \underline{v}_h}{\partial z} + \underline{v}_h\,\frac{\partial Z}{\partial z} - (\underline{v}_h\cdot\nabla)Z\underline{k} - Z(\nabla\cdot\underline{v}_h)\underline{k}$

$$= \frac{\partial}{\partial z}(Z\underline{v}_h) - \underline{k}\,\nabla\cdot(Z\,\underline{v}_h).$$

Also $\int_A \nabla\cdot(Z\,\underline{v}_h)\,dA = A\,\widetilde{\nabla\cdot(Z\,\underline{v}_h)}$ and $C = A\tilde{Z}$.

Thus the vertical component of (A1) may be expressed as

$$\frac{D_f C}{Dt} = A\,\frac{\partial\tilde{Z}}{\partial t} + A\,\hat{\omega}\,\frac{\partial\tilde{Z}}{\partial p} + A\,\widetilde{\nabla\cdot(Z\,\underline{v}_h)} + A\underline{k}\cdot\widetilde{\nabla\omega}\times\frac{\partial\tilde{\underline{v}}}{\partial p}$$

and Eq. (2) simply states that, ignoring $\tilde{F}$, $\frac{D_f C}{Dt} = 0$ in an inertial frame of reference, which is the circulation theorem.